Center of a finite dimensional quantum group

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Dedicated to the dead in the aircrash of MU5735 on March 21st, 2022 in China.

Introduction

The notion of center is abstracted from the center of an algebra by Lurie in *Higher Algebra*. Since then and before the abstraction, it plays an important role both in mathematics and physics:

- one solution of Deligne's conjecture on Hochschild cohomology (Kontsevich, Lurie), tensor category theory and other higher representation theory, open and closed topological quantum field theory (TQFT), ...
- boundary-bulk relation in topological orders (Kong-Wen-Zheng): the suitable algebraic data describing the defects in the bulk of a topological order is always the center of the suitable algebraic data describing the defects on any boundary of the topological order.

Introduction (cont'd)

On the other hand, the study of quantum groups or other quantum algebras, some of them originally arising as deformations of their "classical" counterpart, have deep applications in TQFT, quantum integrable systems, operator algebras, non-commutative geometries, etc., and are interesting subjects of their own rights. Moreover, this field also benefits on and from physics, for example through the study of conformal field theory (for e.g. Dijkgraaf-Pasquier-Roche) , algebraic QFT (for e.g. Mack-Schomerus) and lattice constructions of topological orders (for e.g. Kitaev, Buercschaper et al, Hu-Wan-Wu, Jia et al).

In this talk I introduce the notion of center, and then I focus on quasi-bialgebras. Finally, I announce a result that the 2-centers of a special kind of quasi-bialgebras, finite dimensional Hopf algebras, coincide with the *Drinfeld double* construction of the Hopf algebras.

Centers

We fix a field $\Bbbk$. Let A be a $\Bbbk$-algebra. The center of A is defined to be the subalgebra

$$Z(A) := \{z \in A \mid az = za, \forall a \in A\}.$$

It is observed, for e.g. by Lurie, that the center can be completely characterized by that

- $Z(A)$ is a $\Bbbk$-algebra; There is an algebra homomorphism $\rho\colon Z(A) \otimes_{\Bbbk} A \to A$ such that $\rho(1_Z \otimes -)$ reads identity, where 1_Z is the unit of $Z(A)$;
- It is universal: given a $\Bbbk$-algebra B and an algebra homomorphism $\lambda\colon B \otimes_{\Bbbk} A \to A$ such that $\lambda(1_B \otimes -)$ reads identity, there then exists uniquely an algebra homomorphism $\underline{\lambda}\colon B \to Z(A)$ such that $\lambda = \rho \circ (\underline{\lambda} \otimes \mathrm{id})$, i.e., all actions on A factor through $Z(A) \otimes A \to A$ uniquely.

$$\begin{tikzcd} B \otimes A \arrow[r, "\underline{\lambda}\otimes \mathrm{id}_A"'] \arrow[rr, bend left, "\lambda"] & Z(A) \otimes A \arrow[r, "\rho_Z"'] & A \end{tikzcd}$$

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Remark. The condition "universal" guarantees that if the center of A exists, it is uniquely determined. Hence one can view such a universal property as a definition.

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Check:

➡ $Z(A)$ is a $\Bbbk$-algebra; $\rho\colon Z(A) \otimes_{\Bbbk} A \to A, z \otimes a \mapsto z \cdot a$ is an algebra homomorphism: $\rho(z_1 \otimes a_1)\rho(z_2 \otimes a_2) = z_1 a_1 z_2 a_2 = z_1 z_2 a_1 a_2 = \rho(z_1 z_2 \otimes a_1 a_2)$; moreover, $\rho(1 \otimes -) = \mathrm{id}\colon A \to A$.

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Check:

➡ (Universal) Suppose $\lambda \colon B \otimes_{\Bbbk} A \to A$ is an algebra homomorphism, then $\underline{\lambda} \colon B \to Z(A) \colon b \mapsto \lambda(b \otimes 1_A)$ is indeed well-defined as $a\lambda(b \otimes 1_A) = \lambda(b \otimes a) = \lambda(b \otimes 1_A)a$ for all $a \in A$. Moreover, $\underline{\lambda}$ is an algebra homomorphism.

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Check:

➡ (Universal cont'd) $\rho \circ (\underline{\lambda} \otimes \mathrm{id}) = \lambda\colon B \otimes_{\Bbbk} A \to A$. Moreover, the algebra homomorphism $\underline{\lambda}$ satisfying this condition is unique. ✓

Formalism

We can formalize the story in previous slides using the notion of a *monoidal category*:

$$
\begin{aligned}
\Bbbk\text{-algebras} &\rightsquigarrow \text{objects in a monoidal category } \mathcal{C} \\
\Bbbk\text{-algebra homomorphism} &\rightsquigarrow \text{morphisms in } \mathcal{C} \\
\otimes_{\Bbbk} &\rightsquigarrow \text{the tensor product } \otimes \text{ in } \mathcal{C} \\
\text{distingshed element } 1_Z \in Z(A) \text{ or } 1_B \in B &\rightsquigarrow \text{a morphism } I \to Z(A) \text{ or a morphism } I \to B \text{ in } \mathcal{C}, \\
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Definition

A **monoidal category** is a category $\mathcal{C}$ with a binary operation $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a 0-ary operation $\hat{I} \colon \{*\} \to \mathcal{C}, * \mapsto I$ equipped coherently with natural isomorphisms

- $(X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ for all $X, Y, Z \in \mathcal{C}$;
- $X \otimes I \xrightarrow{\sim} X \xrightarrow{\sim} I \otimes X$ for all $X \in \mathcal{C}$.

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(2)Let $(C, \cdot, 1_C)$ be a monoid. Then C can be viewed as a monoidal category with only trivial morphisms, where the tensor product is given by $X \otimes Y := X \cdot Y$.
(3)The category $\mathrm{LMod}(H)$ of finite dimensional left modules over a Hopf algebra H is a monoidal category. In particular, the category $\mathsf{Rep}(G)$ of representations over a finite group G is a monoidal category. The category $(\mathrm{Vec}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk)$ of $\Bbbk$-vector spaces with tensor product being relative tensor products over $\Bbbk$ is a monoidal category.

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Center: definition

Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and $A \in \mathcal{C}$ be an object. The **center** of A is a triple $(Z(A) \in \mathcal{C}, u_Z \colon I \to Z(A), \rho_Z \colon Z(A) \otimes A \to A)$ where u_Z, ρ are morphisms in $\mathcal{C}$ such that

(I) the diagram
$$\begin{tikzcd}
 & I \otimes A \arrow[dl, "u_Z \otimes \mathrm{id}_A"'] \arrow[dr, "\sim"] & \\
Z(A) \otimes A \arrow[rr, "\rho_Z"] & & A
\end{tikzcd}$$
commutes;

(II) Given any triple $(B, u_B \colon I \to B, \lambda \colon B \otimes A \to A)$ satisfying condition (I) above, there exists a unique morphism $\underline{\lambda} \colon B \to Z(A)$ compatible with u_Z and u_B such that $\lambda = \rho_Z \circ (\underline{\lambda} \otimes \mathrm{id})$,

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Example. In the monoidal category $\mathrm{Alg}(\mathrm{Vec}_{\mathbb{k}})$ of $\mathbb{k}$-algebras, the center of A recovers the usual center of A.

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Example. In the monoidal category $\mathrm{Vec}_{\Bbbk}$ of $\Bbbk$-spaces, the center of $V \in \mathrm{Vec}_{\Bbbk}$ is the space $\mathrm{End}(V)$ equipped with the canonical evaluation map $\mathrm{ev} \colon \mathrm{End}(V) \otimes_{\Bbbk} V \to V, f \otimes v \mapsto f(v)$.

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commutes.

Remark. Suppose the center $(Z(A), Z(A) \otimes A \xrightarrow{\rho} A)$ exists. Then the action $Z(A) \otimes Z(A) \otimes A \xrightarrow{1 \otimes \rho} Z(A) \otimes A \xrightarrow{\rho} A$ induces a morphism $Z(A) \otimes Z(A) \to Z(A)$.

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Remark (cont'd). This makes $Z(A)$ an *algebra* in the monoidal category, and the action on A is a *module action* which is also universal.

2-categorical center

To define center of Hopf algebra we need to consider a categorifed center.

A quick recap on 2-category: In addition to some data resembling those of a 1-category, a 2-category has **2-morphisms between 1-morphisms.** Like the case for 1-morphisms, 2-morphisms can compose with 2-morphisms once their domains and the codomains match in a certain way.

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Remark. Lurie originally defines center in an $(\infty, 1)$-setting, hence in a certain sense his definition automaticaly includes the case for 2-categorical centers.
For example, $\{$categories, functors, natural transformations$\}$ is a 2-category, and $\{$points in a topological space, paths, homotopy classes of homotopies between paths$\}$ is a 2-category.

2-categorical center (cont'd)

Let $(\mathcal{C}, \otimes, I)$ be a monoidal 2-category and $A \in \mathcal{C}$ be an object. The **(2-)center**[1] of A is a quadruple $(Z(A) \in \mathcal{C}, u_Z \colon I \to Z(A), \rho_Z \colon Z(A) \otimes A \to A, \eta)$ where u_Z, ρ are 1-morphisms in $\mathcal{C}$, and η is an invertible 2-morphism in $\mathcal{C}$ shown in the diagram (I).

(I)
$$\begin{tikzcd} & I \otimes A \arrow[dl, "u_Z \otimes \mathrm{id}_A"'] \arrow[dr, "\sim"] \arrow[d, phantom, "\eta \Swarrow"] & \\ Z(A) \otimes A \arrow[rr, "\rho_Z"] & & A \end{tikzcd}.$$

(II) Given any quadruple $(B, u_B \colon I \to B, \lambda \colon B \otimes A \to A, \kappa)$, there exists a 1-morphism $\underline{\lambda} \colon B \to Z(A)$ compatible with u_Z and u_B, which is uniquely up to a specified isomorphism such that the diagram
$$\begin{tikzcd} B \otimes A \arrow[r, "\underline{\lambda} \otimes \mathrm{id}_A"'] \arrow[rr, bend left, "\lambda"] & Z(A) \otimes A \arrow[r, "\rho_Z"'] & A \end{tikzcd}$$
commutes up to an invertible 2-cell compatible with η and κ.

[1] In fact, this definition is weaker than the real definition of 2-center using representable 2-functors. However, we ignore the differences here.

2-categorical center (cont'd)

The taka-away: the center of A in a monoidal 2-category $(\mathcal{C}, \boxtimes, \mathcal{J})$ is still equipped with a universal action $Z(A) \boxtimes A \to A$.

[2] Note that the universal action is a monoidal functor, and the monoidal structure makes use of γ.

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Examples. (1)In the monoidal 2-category $(\mathrm{Cat}, \times, \{*\})$ of categories, functors and natural transformations, the center of a category $\mathcal{A} \in \mathrm{Cat}$ is the category $\mathrm{End}(\mathcal{A})$ of functors $\mathcal{A} \to \mathcal{A}$ equipped with the canonical evaluation functor $\mathrm{ev}\colon \mathrm{End}(\mathcal{A}) \times \mathcal{A} \to \mathcal{A}, (F, A) \mapsto F(A)$.

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(2)In the 2-category MonCat of monoidal categories, monoidal functors and monoidal natural transformations, the center $Z(\mathcal{A})$ of $(\mathcal{A}, \otimes, I) \in \mathrm{MonCat}$ is called the **Drinfeld center**, and is a monoidal category whose set of objects are

$$Z(\mathcal{A}) := \{(Z \in \mathcal{A}, \gamma_{-,z})\},$$

where $\gamma_{-,Z} = \{\gamma_{X,Z}\colon X \otimes Z \xrightarrow{\sim} Z \otimes X\}_{X \in \mathcal{A}}$ is a set of isomorphisms natural in X and satisfying certain coherence relations. The universal action reads

$$Z(\mathcal{A}) \times \mathcal{A} \to \mathcal{A}, ((Z, \gamma_{-,Z}), A) \mapsto Z \otimes A.^{2}$$

[2] Note that the universal action is a monoidal functor, and the monoidal structure makes use of γ.

Drinfeld centers appear in both Kong and Zhao's talk on Tuesday, and indeed it plays an important role in tensor category theory and physics. Let us see some examples.

Examples of Drinfeld centers. (1)Suppose C is the monoidal category obtained from a monoid $(C, \cdot, 1_C)$ by adding trivial morphisms. Then the Drinfeld center of C is the center of the monoid C (with trivial morphisms.)
(2)Let G be a finite group. Then the Drinfeld center $Z(\text{Rep}(G))$ of the representation category of G is the category of G-graded G-representations in which the grading respects the G-action in a certain way.
(3)Let H be a finite dimensional Hopf algebra. Then the Drinfeld center $Z(\mathrm{LMod}(H))$ is the representation category of the Drinfeld double $D(H)$ of H.
(4)The suitable algebraic description of the (2+1)-dimensional bulk topological order is the Drinfeld center of the algebraic description of the boundary (1+1)-dimensional topological order. (Kong:1307.8244, Kitaev-Kong:1104.5047).

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Remark. Drinfeld center of a monoidal category $\mathcal{A}$ is in general not a subcategory of $\mathcal{A}$.

Short summary

We have seen:

- Center of an object in a monoidal category. For e.g., center of a $\mathbb{k}$-algebra, center of a vector space.
- Center of an object in a monoidal 2-category. For e.g., center of a monoidal category.

Center of a Hopf algebra

We do not present the proof of our main result computing the center of finite dimensional Hopf algebras. My aim is to state the result and show why we can expect such a result is true. The rest of my talk is mostly devoted to make the necessary prepration.

- Preparation: Tannaka-Krein duality and quasi-bialgebras.
- Stating the main result and show how it can be expected.

Tannaka-Krein duality

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$$A \xrightarrow{\phi} B \ : \ \text{algebra homomorphism} \quad \xmapsto{\mathrm{TK}} \quad \mathrm{LMod}(A) \xleftarrow{\phi^*} \mathrm{LMod}(B) \ : \ \text{functor}$$

Tannaka-Krein duality

Tannaka-Krein duality deals with the relation between algebras and its category of left modules/representations. From now on we assume finite dimensionality for both algebras and their modules.

$$
\begin{tikzcd}
A \arrow[r, bend left=50, "\phi"] \arrow[r, bend right=50, "\psi"'] \arrow[r, phantom, "b\Downarrow"] & B
\end{tikzcd}
\;:\; \text{“intertwinner”} \quad \overset{\mathrm{TK}}{\longmapsto} \quad
\begin{tikzcd}
\mathrm{LMod}(A) \arrow[r, phantom, "b^*\Downarrow"] & \mathrm{LMod}(B) \arrow[l, bend right=30, "\phi^*"'] \arrow[l, bend left=30, "\psi^*"]
\end{tikzcd}
\;:\; \text{natural transformation}
$$

Here an “intertwinner” $b\colon \phi \Rightarrow \psi$ between algebra homomorphisms is an element $b \in B$ such that $b\phi(a) = \psi(a)b$ for all $a \in A$. Then the component of the natural transformation b^* reads $b.-\colon \phi^*({}_BV) \to \psi^*({}_BV)$ for ${}_BV \in \mathrm{LMod}(B)$.

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\begin{tikzcd} A \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "b\Downarrow"] & B \end{tikzcd} : \text{"intertwinner"} \quad \xmapsto{\mathrm{TK}} \quad \begin{tikzcd} \mathrm{LMod}(A) \arrow[r, phantom, "b^*\Downarrow"] & \mathrm{LMod}(B) \arrow[l, bend right, "\phi^*"'] \arrow[l, bend left, "\psi^*"] \end{tikzcd} : \text{natural transformation}
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Remark. Note that on this last level the action of TK is *fully-faithful*, in the sense that all natural transformations $\alpha\colon \phi^* \Rightarrow \psi^*$ arise in this way for a unique $b \in B$.

Exercise for TK-duality: Equip $\mathrm{LMod}(H)$ with a monoidal structure!

Exercise! Suppose H is an algebra. How to equip $\mathrm{LMod}(H)$ with a monoidal structure assuming you know TK duality?

Functor $\otimes\colon \mathrm{LMod}(H)\times\mathrm{LMod}(H)\to\mathrm{LMod}(H\otimes_{\Bbbk}H)\to\mathrm{LMod}(H)$

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$$\text{Algebra homomorphism } \Bbbk \leftarrow H : \varepsilon \quad \longmapsto \quad \text{Functor } \hat{I} \colon \{*\} \to \mathrm{LMod}(\Bbbk) \to \mathrm{LMod}(H)$$

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Natural isomorphism

$$\begin{tikzcd}
\mathrm{LMod}(H)\times\mathrm{LMod}(H)\times\mathrm{LMod}(H) \arrow[r, "\otimes\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\otimes"'] \arrow[dr, phantom, "\alpha\Swarrow"] & \mathrm{LMod}(H)\times\mathrm{LMod}(H) \arrow[d, "\otimes"] \\
\mathrm{LMod}(H)\times\mathrm{LMod}(H) \arrow[r, "\otimes"'] & \mathrm{LMod}(H)
\end{tikzcd}$$

Exercise for TK-duality: Equip $\mathrm{LMod}(H)$ with a monoidal structure!

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$$
\begin{aligned}
\text{Algebra homomorphism } H \otimes_{\Bbbk} H \leftarrow H : \Delta \quad &\longmapsto \quad \text{Functor } \otimes : \mathrm{LMod}(H) \times \mathrm{LMod}(H) \to \mathrm{LMod}(H \otimes_{\Bbbk} H) \to \mathrm{LMod}(H) \\
\text{Algebra homomorphism } \Bbbk \leftarrow H : \varepsilon \quad &\longmapsto \quad \text{Functor } \hat{I} : \{*\} \to \mathrm{LMod}(\Bbbk) \to \mathrm{LMod}(H) \\
\text{Invertible intertwinner} \quad &\longmapsto \quad \text{Natural isomorphism}
\end{aligned}
$$

$$
\begin{tikzcd}
H \otimes_{\Bbbk} H \otimes_{\Bbbk} H & H \otimes_{\Bbbk} H \arrow[l, "\Delta \otimes \mathrm{id}"'] \\
H \otimes_{\Bbbk} H \arrow[u, "\mathrm{id} \otimes \Delta"] & H \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \arrow[ul, phantom, "a \Swarrow"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{LMod}(H) \times \mathrm{LMod}(H) \times \mathrm{LMod}(H) \arrow[r, "\otimes \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \otimes"'] \arrow[dr, phantom, "\alpha \Swarrow"] & \mathrm{LMod}(H) \times \mathrm{LMod}(H) \arrow[d, "\otimes"] \\
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$$

Hopf algebras

Summarising the structures we get on H, we have:

Definition. A **quasi-bialgebra** is an algebra H equipped with algebra homomorphisms $\Delta\colon H \to H \otimes H, \varepsilon\colon H \to \Bbbk$, and an invertible interwinner $a \in H \otimes H \otimes H$ in the sense that

$$a \cdot (\Delta \otimes \mathrm{id})\Delta(h) = (\mathrm{id} \otimes \Delta)\Delta(h) \cdot a, \forall h \in H,$$

subject to conditions

$$(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta\colon H \to H$$
$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(a) \cdot (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(a) = (1_H \otimes a) \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(a) \cdot (a \otimes 1_H)$$
$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(a) = 1_H \otimes 1_H.$$

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$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(a) = 1_H \otimes 1_H.$$

Proposition. If H is a quasi-bialgebra, then $\mathrm{LMod}(H)$ is a monoidal category.

Definition. A **bialgebra** is a quasi-bialgebra H with $a = 1_H \otimes 1_H \otimes 1_H$. In particular, in a bialgebra, (H, Δ, ε) always form a coassociative coalgebra. A **Hopf algebra** is a bialgebra which admits an "antipode".

Let H, K be quasi-bialgebras.

Functor $F\colon \mathrm{LMod}(H) \to \mathrm{LMod}(K)$

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Algebra homomorphism $H \leftarrow K$ $\longmapsto$ Functor $F\colon \mathrm{LMod}(H) \to \mathrm{LMod}(K)$

A monoidal structure on F:

$$\begin{tikzcd}
\mathrm{LMod}(H)\times\mathrm{LMod}(H) \arrow[r] \arrow[d, "F\times F"'] \arrow[dr, phantom, "\Nearrow T_2"] & \mathrm{LMod}(H) \arrow[d, "F"] \\
\mathrm{LMod}(K)\times\mathrm{LMod}(K) \arrow[r] & \mathrm{LMod}(K)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\{*\} \arrow[r] \arrow[dr] & \mathrm{LMod}(H) \arrow[d, "F"] \arrow[l, phantom, shift right=6, "\Nearrow T_0"] \\
& \mathrm{LMod}(K)
\end{tikzcd}$$

Let H, K be quasi-bialgebras.

$$\text{Algebra homomorphism } H \leftarrow K \quad \longmapsto \quad \text{Functor } F\colon \mathrm{LMod}(H) \to \mathrm{LMod}(K)$$

$$\text{Interwinners} \quad \longmapsto \quad \text{A monoidal structure on } F\text{:}$$

$$\begin{tikzcd} H\otimes_{\Bbbk} H & H \arrow[l, "\Delta_H"'] \\ K\otimes_{\Bbbk} K \arrow[u, "f\otimes f"] \arrow[ur, phantom, "\nearrow\!\!\!\!\nearrow\ t_2"] & K \arrow[l, "\Delta_K"] \arrow[u, "f"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{LMod}(H)\times\mathrm{LMod}(H) \arrow[r] \arrow[d, "F\times F"'] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow\ T_2"] & \mathrm{LMod}(H) \arrow[d, "F"] \\ \mathrm{LMod}(K)\times\mathrm{LMod}(K) \arrow[r] & \mathrm{LMod}(K) \end{tikzcd}$$

$$\text{and} \quad \begin{tikzcd} \Bbbk & H \arrow[l, "\varepsilon_H"'] \\ & K \arrow[ul, "\varepsilon_K"] \arrow[u, "f"'] \arrow[ul, phantom, shift right=3, "\nearrow\!\!\!\!\nearrow\ t_0"] \end{tikzcd} \qquad \text{and} \quad \begin{tikzcd} \{*\} \arrow[r] \arrow[dr] & \mathrm{LMod}(H) \arrow[d, "F"] \arrow[dl, phantom, "\nearrow\!\!\!\!\nearrow\ T_0"] \\ & \mathrm{LMod}(K) \end{tikzcd}$$

satisfying certain conditions.

Definition. Let H, K be quasi-bialgebras. A **quasi-bialgebra homomorphism** $H \to K$ is an algebra homomorphism $f\colon K \to H$ equipped with invertible intertwinners $t_2 \in H \otimes_{\mathbb{k}} H$ and $t_0 \in \mathbb{k}$ in the sense that

$$\begin{aligned} t_2 \cdot (f \otimes f)\Delta_K(k) = \Delta_H(f(k)) \cdot t_2, \forall k \in K \\ t_0 \cdot \varepsilon_K(k) = \varepsilon_H(f(k)) \cdot t_0, \forall k \in K, \end{aligned}$$

subject to conditions

$$\begin{gathered} (\Delta_H \otimes \mathrm{id})(t_2) \cdot (t_2 \otimes 1_H) = (\mathrm{id} \otimes \Delta_H)(t_2) \cdot (1_H \otimes t_2) \\ t_0(\varepsilon_H \otimes \mathrm{id})(t_2) = 1_H = t_0(\mathrm{id} \otimes \varepsilon_H)(t_2). \end{gathered}$$

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subject to conditions

$$(\Delta_H \otimes \mathrm{id})(t_2) \cdot (t_2 \otimes 1_H) = (\mathrm{id} \otimes \Delta_H)(t_2) \cdot (1_H \otimes t_2)$$
$$t_0(\varepsilon_H \otimes \mathrm{id})(t_2) = 1_H = t_0(\mathrm{id} \otimes \varepsilon_H)(t_2).$$

Definition. Let $(f, t_2, t_0), (g, s_2, s_0)\colon H \to K$ be quasi-bialgebra homomorphisms. A **quasi-bialgebra 2-homomorphism** $(f, t) \Rightarrow (g, s)$ is an intertwinner $\eta\colon f \Rightarrow g \in H$ such that

$$\Delta_H(\eta) \cdot t_2 = s_2 \cdot (\eta \otimes \eta), \quad \varepsilon_H(\eta) t_0 = s_0.$$

Then we have:

$$\text{quasi-bialgebra homomorphisms} \xmapsto{\mathrm{TK}} \text{monoidal functors}$$

$$\text{quasi-bialgebra 2-homomorphisms} \xmapsto{\mathrm{TK}} \text{monoidal natural transformations.}$$

Then we have:

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Fact (McCrudden)

Quasi-bialgebras, quasi-bialgebra homomorphisms, quasi-bialgebra 2-homomorphisms form a 2-category.

Moreover, this 2-category, denoted by QB, is a monoidal 2-category under $\otimes_{\Bbbk}$.

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Moreover, this 2-category, denoted by QB, is a monoidal 2-category under $\otimes_{\mathbb{k}}$.

Main Theorem

Finite dimensional Hopf algebras admits center in QB. The center of a Hopf algebra H is given by the *Drinfeld double* of H.

Drinfeld double

Let H be a finite dimensional Hopf algebra.

- Its antipode $S\colon H \to H$ is an invertible algebra anti-homomorphism and also a coalgebra-antihomomorphism satisfying certain relations.
- Note that the dual space $H^* := \mathrm{Hom}_{\Bbbk}(H, \Bbbk)$ has a natural Hopf algebra structure; we use (H^{op}) to refer to the same Hopf algebra H with multiplication reversed.

The **Drinfeld double** $D(H)$ (Drinfeld, 1987) of H is the bialgebra whose underlying coalgebra is

$$(H^{\mathrm{op}})^* \otimes_{\Bbbk} H,$$

and whose multiplication is given by

$$(f \otimes a) \cdot (g \otimes b) := f \cdot g(S^{-1}(a_{(3)}) - a_{(1)}) \otimes a_{(2)} b.$$

Fact. $D(H)$ is a Hopf algebra, and $\mathrm{LMod}(D(H)) \cong Z(\mathrm{LMod}(H))$.

Main theorem revisited

$$
\begin{tikzcd}
 & \text{Hopf algebras} \arrow[r, "\mathrm{TK}"] & \text{Monoidal categories} \arrow[d, "\text{taking center}"'] \\
 & \text{Drinfeld double of Hopf algebras} \arrow[r, "\mathrm{TK}"'] & \text{Monoidal categories}
\end{tikzcd}
$$

Main theorem revisited

$$
\begin{tikzcd}
\text{Hopf algebras} \arrow[r, "\mathrm{TK}"] \arrow[d, dashed, "\text{my main theorem: }"', "\text{taking centers}"] & \text{Monoidal categories} \arrow[d, "\text{taking center}"'] \\
\text{Drinfeld double of Hopf algebras} \arrow[r, "\mathrm{TK}"'] & \text{Monoidal categories}
\end{tikzcd}
$$

Main theorem revisited

$$\begin{tikzcd}
\text{Hopf algebras} \arrow[r, "\mathrm{TK}"] \arrow[d, dashed, "\text{my main theorem: }"', "\text{taking centers}"] & \text{Monoidal categories} \arrow[d, "\text{taking center}"'] \\
\text{Drinfeld double of Hopf algebras} \arrow[r, "\mathrm{TK}"'] & \text{Monoidal categories}
\end{tikzcd}$$

The main theorem (cont'd)

The universal action $D(H) \otimes_{\Bbbk} H \to H$ is given by the quasi-bialgebra homomorphism $(f : H \to D(H) \otimes_{\Bbbk} H, t_2, t_0)$ where

$$f : h \mapsto \hat{1} \otimes h_{(1)} \otimes h_{(2)},$$

$$t_2 = \sum_i \hat{1} \otimes 1_H \otimes e_i \otimes e^i \otimes 1_H \otimes 1_H, \quad , t_0 = 1.$$

Here $\hat{1}$ is the unit of H^*, and $\{e_i\}_i$ is a basis of H with $\{e^i\}_i$ being its dual basis.

Acknowledgements. I thank Liang Kong and Zhi-Hao Zhang for letting me know centers and the latter also for communicating me the Drinfeld center of Rep(G).

Further readings.

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Thank you for your attention! Questions and comments are welcome.