The algebraic structure of elliptic quantum groups

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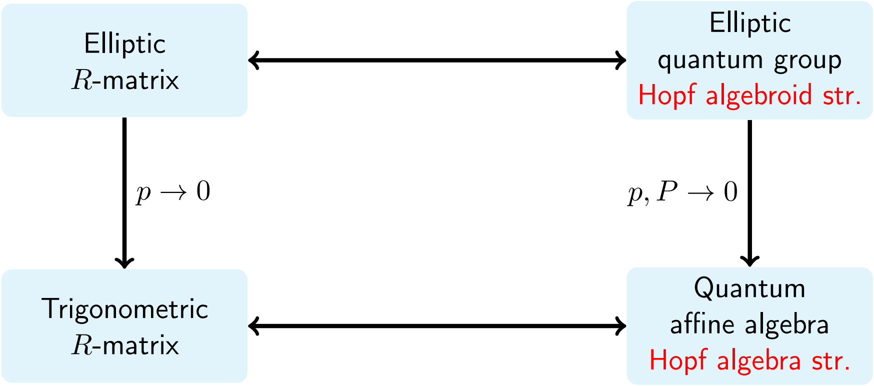
Conceptual diagram



R-matrix: A solution of the Yang-Baxter equation; $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.

1 Drinfeld realization of quantum affine algebras

Quantum affine algebras:

Hopf algebras associating with affine root systems

$\to$ Applications to Yang-Baxter eq., solvable lattice models etc.

$A = (a_{ij})_{i,j=0}^{l}$: GCM of a non-twisted affine root system $X_l^{(1)}$.

Drinfeld realization of quantum affine algebras [Dr89]

- gens: $x_{i,j}^{\pm}$, h_{ik}, $K_i^{\pm}$, $C^{\pm 1/2}$, D $(1 \le i \le l,\ j \in \mathbb{Z},\ k \in \mathbb{Z} \setminus \{0\})$.
- rels: $C^{\pm 1/2}$: center.

$$[K_j, h_{ik}] = [K_i, K_j] = 0, \quad K_i x_{jk}^{\pm} K_i^{-1} = q^{\pm(\alpha_i \mid \alpha_j)} x_{jk}^{\pm},$$

$$[h_{ik}, h_{jl}] = \delta_{k+l,0} \frac{[k a_{ij}]_i}{k} \frac{C^k - C^{-k}}{q_j - q_j^{-1}},$$

$$[h_{ik}, x_{jl}^{\pm}] = \pm \frac{[k a_{ij}]_i}{k} C^{\mp(|k|/2)} x_{j,k+l}^{\pm}, \text{ etc.}$$

$\exists$ Much simpler presentations by currents (generating functions of generators).
The current relations are given by structure functions $g_{ij}^{\mathrm{aff}}(z)$.

Currents and relations

$$x_i^{\pm}(z) := \sum_{n\in\mathbb{Z}} x_{in}^{\pm} z^{-n},$$

$$\psi_i(z) = \sum_{k\geq 0} \psi_{ik} z^{-k} := K_i \exp\Big((q_i - q_i^{-1}) \sum_{k>0} h_{ik} z^{-k}\Big),$$

$$\varphi_i(z) = \sum_{k\geq 0} \varphi_{ik} z^{k} := K_i^{-1} \exp\Big(-(q_i - q_i^{-1}) \sum_{k>0} h_{i,-k} z^{k}\Big).$$

$$\psi_i(z)\psi_j(w) = \psi_j(w)\psi_i(z), \quad \varphi_i(z)\varphi_j(w) = \varphi_j(w)\varphi_i(z),$$

$$\varphi_i(z)\psi_j(w) = g_{ij}^{\mathrm{aff}}(C^{-1}z/w)(g_{ij}^{\mathrm{aff}}(Cz/w))^{-1}\psi_j(w)\varphi_i(z),$$

$$\varphi_i(z)x_j^{\pm}(w) = \Big(g_{ij}^{\mathrm{aff}}(C^{\mp 1/2}z/w)\Big)^{\pm 1} x_j^{\pm}(w)\varphi_i(z), \text{etc.}$$

1 Drinfeld realization of quantum affine algebras

Structure functions

$$g_{ij}^{\mathsf{aff}}(z) := q_i^{-a_{ij}} \frac{1 - q_i^{a_{ij}} z}{1 - q_i^{-a_{ij}} z} \quad (i, j = 1, \dots, l)$$

$$g_{ij}^{\mathsf{aff}}(z^{-1})^{-1} = g_{ji}^{\mathsf{aff}}(z).$$

Drinfeld comultiplication

$$\Delta(C^{\pm 1/2}) = C^{\pm 1/2} \otimes C^{\pm 1/2}, \quad \Delta(D) = D \otimes D,$$

$$\Delta(x_i^+(z)) = x_i^+(z) \otimes 1 + \varphi_i(C_{(1)}^{1/2} z) \otimes x_i^+(C_{(1)} z),$$

$$\Delta(x_i^-(z)) = 1 \otimes x_i^-(z) + x_i^-(C_{(2)} z) \otimes \psi_i(C_{(2)}^{1/2} z),$$

$$\Delta(\varphi_i(z)) = \varphi_i(C_{(2)}^{-1/2} z) \otimes \varphi_i(C_{(1)}^{1/2} z), \quad \Delta(\psi_i(z)) = \psi_i(C_{(2)}^{1/2} z) \otimes \psi_i(C_{(1)}^{-1/2} z).$$

2.1 Elliptic quantum groups

A dynamical-elliptic analogue of Drinfeld realizations of q-affine algebras of type $X_l^{(1)}$.

Elliptic quantum group [JKOS99]

An H-Hopf algebroid (to be explained in §3.2) $U_{q,p}(X_l^{(1)})$ with

- parameters: $q \in \mathbb{C}$ $(0 < |q| < 1)$, elliptic norm p,
  dynamical parameters P_i $(i = 1, \ldots, l)$,
- ground ring: $\mathbb{C}[\![p]\!]$,
- structure functions: theta functions

$$g_{ij}^{\mathsf{ell}}(x;p) = \frac{G_{ij}^{+}(x;p)}{G_{ij}^{-}(x;p)} := \frac{q_i^{-a_{ij}}\theta(q_i^{a_{ij}}x;p)}{\theta(q_i^{-a_{ij}}x;p)},$$
$$\theta(x;p) := (x, px^{-1};p)_\infty \in \mathbb{C}[x^{\pm 1}][\![p]\!],$$

- relations: dynamical analogue of quantum affine algebra.

2.1 Elliptic quantum groups

Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of a finite root system X_l. i.e.
$\Pi := \{\alpha_1, \ldots, \alpha_l\} \subset \mathfrak{h}^*, \Pi^\vee := \{\alpha_1^\vee, \ldots, \alpha_l^\vee\} \subset \mathfrak{h}, \langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$.

Dynamical parameter

For dynamical parameters P_i $(i = 1, \ldots, l)$, set

$$\widehat{H} := \bigoplus_{i=1}^{l} \mathbb{C}(P_i + \alpha_i^\vee) \oplus \left(\bigoplus_{i=1}^{l} \mathbb{C}P_i\right).$$

Denote the field of meromorphic functions of $\widehat{H}^*$ by $\mathcal{M}(\widehat{H}^*)$.

Properties of structure functions

p: formal parameter, $\{G_{ij}^{\pm}(z;p) \mid i,j \in I\}$: A set of functions satisfying the following Ding-Iohara conditions.

- $G_{ij}^{\pm}(z;p) \in \mathbb{C}[z^{\pm 1}][\![p]\!]$ and invertible in $\mathbb{C}(\!(z)\!)[\![p]\!]$.
- The formal power series $g_{ij}(z;p) := \frac{G_{ij}^{+}(z;p)}{G_{ij}^{-}(z;p)} \in \mathbb{C}(\!(z)\!)[\![p]\!]$ satisfies
$$g_{ij}(z^{-1};p) = g_{ji}(z;p)^{-1}.$$

Elliptic quantum group $U_{q,p}(X_l^{(1)})$

- gens.: $\mathcal{M}(\widehat{H}^*)$, $q^{\pm c/2}$, d, $K_i^{\pm}$, $e_{i,m}$, $f_{i,m}$, $\alpha^\vee_{i,n}$ $\quad (i \in I,\ m \in \mathbb{Z},\ n \in \mathbb{Z} \setminus \{0\})$
  Introduce the currents as follows.

$$e_i(z) := \sum_{m \in \mathbb{Z}} e_{i,m} z^{-m}, \quad f_i(z) := \sum_{m \in \mathbb{Z}} f_{i,m} z^{-m},$$

$$\psi_i^+(q^{-\frac{c}{2}} z) := K_i^+ \exp\Big(-(q_i - q_i^{-1}) \sum_{n>0} \frac{\alpha^\vee_{i,-n}}{1-p^n} z^n\Big) \exp\Big((q_i - q_i^{-1}) \sum_{n>0} \frac{p^n \alpha^\vee_{i,n}}{1-p^n} z^{-n}\Big).$$

- rels: $q^{\pm c/2}$: center,

$$\psi_i^{\pm}(z)\psi_j^{\pm}(w) = \frac{g^*_{ij}(\frac{z}{w})}{g_{ij}(\frac{z}{w})} \psi_j^{\pm}(w)\psi_i^{\pm}(z), \quad \psi_i^+(z)\psi_j^-(w) = \frac{g^*_{ij}(q^{-c}\frac{z}{w})}{g_{ij}(q^{c}\frac{z}{w})} \psi_j^-(w)\psi_i^+(z),$$

$$\psi_i^+(z)e_j(w) = g^*_{ij}(q^{-c/2}\tfrac{z}{w}) e_j(w)\psi_i^+(z),\ \psi_i^+(z)f_j(w) = g_{ij}(q^{c/2}\tfrac{z}{w})^{-1} f_j(w)\psi_i^+(z) \text{ etc.}$$

2.2 Drinfeld Comultiplication

Elliptic quantum group $U_{q,p}(X_l^{(1)})$ admits an H-Hopf algebroid structure, having Drinfeld comultiplication with modified tensor product $\widetilde{\otimes}$.

Drinfeld comultiplication of $U_{q,p}(X_l^{(1)})$

$$\Delta(q^{\pm c/2}) := q^{\pm c/2} \,\widetilde{\otimes}\, q^{\pm c/2}, \quad \Delta(d) := d \,\widetilde{\otimes}\, 1 + 1 \,\widetilde{\otimes}\, d,$$
$$\Delta(K_i^{\pm}) := K_i^{\pm} \,\widetilde{\otimes}\, K_i^{\pm},$$
$$\Delta(e_i(z)) := e_i(z) \,\widetilde{\otimes}\, 1 + \psi_i^{+}(q^{c_1/2}z) \,\widetilde{\otimes}\, e_i(q^{c_1}z),$$
$$\Delta(f_i(z)) := 1 \,\widetilde{\otimes}\, f_i(z) + f_i(q^{c_2}z) \,\widetilde{\otimes}\, \psi_i^{-}(q^{c_2/2}z),$$
$$\Delta(\psi_i^{+}(z)) := \psi_i^{+}(q^{-c_2/2}z) \,\widetilde{\otimes}\, \psi_i^{+}(q^{c_1/2}z),$$
$$\Delta(\psi_i^{-}(z)) := \psi_i^{-}(q^{c_2/2}z) \,\widetilde{\otimes}\, \psi_i^{-}(q^{-c_1/2}z).$$

H-Hopf algebroid: A bimodule analogue of a Hopf algebra[Etingof, Varchenko 1998].
H: finite dimensional $\mathbb{C}$-linear space,

$\mathbb{F}$: the field of meromorphic functions on H^*.

H-prealgebras

A tuple $A := (\bigoplus_{\alpha,\beta} A_{\alpha,\beta}, \sigma, \tau)$ is an H-prealgebra if

- $A = \bigoplus_{\alpha,\beta} A_{\alpha,\beta}$ is an $H^* \times H^*$-graded $\mathbb{C}$-linear space,
- (A, σ) and (A, τ) are graded $\mathbb{F}$-bimodules,
- the images of σ and τ commute,
- for any $f \in \mathbb{F}$ and $a \in A_{\alpha,\beta}$

$$\sigma(f)a = a\sigma(T_\alpha f), \quad \tau(T_\beta f)a = a\tau(f),$$

  where $T_\alpha(f)(x) = f(x+\alpha)$ for $f \in \mathbb{F}$, $\alpha, x \in H^*$.

Example: Difference operator ring

$D_H := \bigoplus_{\alpha\in H^*} \mathbb{F}T_{-\alpha}$ possesses an H-prealgebra structure s.t.

$$(D_H)_{\alpha,\beta} := \begin{cases} \mathbb{F}T_{-\alpha} & (\alpha=\beta) \\ 0 & (\alpha\neq\beta), \end{cases} \qquad \sigma(f)\cdot = \cdot\tau(f) = f\circ \quad (f\in\mathbb{F}).$$

The monoidal category H-prealg

A tensor product $\widetilde{\otimes}$ in the category of H-prealgebras:

$$A \mathbin{\widetilde{\otimes}} B := \bigoplus_{\alpha,\beta\in H^*}\Big(\bigoplus_{\gamma\in H^*}(A_{\alpha,\gamma}\mathbin{\widetilde{\otimes}} B_{\gamma,\beta})\Big),$$

$$A_{\alpha,\gamma}\mathbin{\widetilde{\otimes}} B_{\gamma,\beta} := A_{\alpha,\gamma}\otimes_{\mathbb{C}} B_{\gamma,\beta}/(a\tau(f)\otimes b - a\otimes\sigma(f)b),$$

$$\sigma(f)(a\mathbin{\widetilde{\otimes}} b)\sigma(g) := (\sigma(f)a\sigma(g)\mathbin{\widetilde{\otimes}} b),$$

$$\tau(f)(a\mathbin{\widetilde{\otimes}} b)\tau(g) := a\mathbin{\widetilde{\otimes}}(\tau(f)b\tau(g)).$$

$\to$ (H-prealg, $\widetilde{\otimes}$, D_H) is a monoidal category.

H-coalgebroid

An H-coalgebroid (A, Δ, ε) is a comonoid object in H-prealg.
A tensor product $\widehat{\otimes}$ in the category of H-coalgebroid:

$$A \mathbin{\widehat{\otimes}} B := \bigoplus_{\alpha,\beta} \Big(\bigoplus_{\gamma,\delta} A_{\gamma,\delta} \mathbin{\widehat{\otimes}} B_{\alpha-\gamma,\beta-\delta} \Big),$$

$$A_{\gamma,\delta} \mathbin{\widehat{\otimes}} B_{\alpha-\gamma,\beta-\delta} := A_{\gamma,\delta} \otimes_{\mathbb{C}} B_{\alpha-\gamma,\beta-\delta} / (\tau(f) a \sigma(g) \otimes b - a \otimes \sigma(g) b \tau(f)),$$

$$\sigma(f)(a \mathbin{\widehat{\otimes}} b)\sigma(g) := (\sigma(f)a) \mathbin{\widehat{\otimes}} (b\sigma(g)),$$

$$\tau(f)(a \mathbin{\widehat{\otimes}} b)\tau(g) := (a\tau(g) \mathbin{\widehat{\otimes}} \tau(f)b).$$

$\rightarrow$ (H-coalg, $\widehat{\otimes}$, $\mathbb{F}$) is a monoidal category. An H-bialgebroid is a monoid in H-coalgd.

H-Hopf algebroid

A:H-bialgebroid. $S\colon A \to A$ is the antipode if

$$S(a\sigma(f)) = S(a)\tau(f), \quad S(a\tau(f)) = S(a)\sigma(f),$$
$$\sum a_{(1)}S(a_{(2)}) = \sigma(\varepsilon(a)\cdot 1)1_A,$$
$$\sum S(a_{(1)})a_{(2)} = 1_A\tau(T_\alpha(\varepsilon(a)\cdot 1)) \quad (a \in A_{\alpha,\beta}),$$

where $\Delta(a) = \sum a_{(1)} \mathbin{\widetilde{\otimes}} a_{(2)}$. H-Hopf algebroid is an H-bialgebroid with the antipode.

Example: Difference operator ring

D_H is an H-Hopf algebroid by

$$\Delta\colon fT_{-\alpha} \mapsto fT_{-\alpha} \mathbin{\widetilde{\otimes}} T_{-\alpha}, \ \varepsilon := \mathrm{id}_{D_H}, \ S\colon fT_{-\alpha} \mapsto (T_\alpha f)T_\alpha.$$

$\to$ Setting $H = 0$, we rocover ordinary Hopf algebras.

$U_{q,p}(X_l^{(1)})$: Elliptic quantum group of type $X_l^{(1)}$.
$H := \bigoplus_{1\le i\le l} \mathbb{C}P_i$: Cartan subalgebra of type X_l.

$\mathbb{F}$-actions on $U_{q,p}(X_l^{(1)})$

$\mathbb{F}$: The field of meromorphic functions on H^*.
$\mu_l, \mu_r : \mathbb{F}[\![p]\!] \to U_{q,p}(X_l^{(1)})_{0,0}$: moment maps

$$\mu_l(f) := f(P, p^*), \quad \mu_r(f) := f(P+h, p).$$

Define $\mathbb{F}$-actions σ and τ by

$$\sigma(f)a\sigma(g) := \mu_l(f)a\mu_l(g), \quad \tau(f)a\tau(g) := \mu_r(g)a\mu_r(f).$$

3.2 H-Hopf algebroid structure of $U_{q,p}(X_l^{(1)})$

Dynamical shifts

$U_{q,p}(X_l^{(1)})$ admits defining relations (so called dynamical shift)

$$g(P)e_i(z) := e_i(z)g(P - \langle P, \alpha_i \rangle), \quad g(P+h)e_i(x) := e_i(z)g(P),$$
$$g(P)f_i(z) := f_i(z)g(P), \quad g(P+h)f_i(z) := f_i(z)g(P - \langle P, \alpha_i \rangle)\text{etc.}$$

In terms of H-Hopf algebroid, this means

$$e_i(z) \in (U_{q,p}(X_l^{(1)}))_{-\alpha_i,0}[\![z]\!], \quad f_i(z) \in (U_{q,p}(X_l^{(1)}))_{0,-\alpha_i}[\![z]\!].$$

Modified tensor $\widetilde{\otimes}$

Modified tensor product $\widetilde{\otimes}$ gives rises to an exchange of dynamical parameters;

$$g(P+h,p)\,\widetilde{\otimes}\,1 = 1\,\widetilde{\otimes}\,g(P,p^*).$$

As an example, we show that the next equation holds.

$$\Delta(\psi_i^+(z))\Delta(\psi_j^+(w)) = \Delta\left(\frac{g_{ij}^*(\frac{z}{w})}{g_{ij}(\frac{z}{w})}\right)\Delta(\psi_j^+(w))\Delta(\psi_i^+(z)),$$

where $g_{ij}(z) \coloneqq g_{ij}(z;p),\ g_{ij}^*(z) \coloneqq g_{ij}(z;p^*)$.

The proof of the above equation

$$\begin{aligned}
\Delta(\psi_i^+(z))\Delta(\psi_j^+(w)) &= \psi_i^+(q^{c_2/2}z)\psi_j^+(q^{c_2/2}w)\,\widetilde{\otimes}\,\psi_i^+(q^{-c_1/2}z)\psi_j^+(q^{-c_1/2}w) \\
&= \frac{g_{ij}^*(\frac{z}{w})}{g_{ij}(\frac{z}{w})}\psi_j^+(q^{c_2/2}w)\psi_i^+(q^{c_2/2}z)\,\widetilde{\otimes}\,\frac{g_{ij}^*(\frac{z}{w})}{g_{ij}(\frac{z}{w})}\psi_j^+(q^{-c_1/2}w)\psi_i^+(q^{-c_1/2}z) \\
&= g_{ij}^*(\tfrac{z}{w})\psi_j^+(q^{c_2/2}w)\psi_i^+(q^{c_2/2}z)\,\widetilde{\otimes}\,\frac{1}{g_{ij}(\frac{z}{w})}\psi_j^+(q^{-c_1/2}w)\psi_i^+(q^{-c_1/2}z) \\
&= \Delta\left(\frac{g_{ij}^*(\frac{z}{w})}{g_{ij}(\frac{z}{w})}\right)\Delta(\psi_j^+(w))\Delta(\psi_i^+(z))
\end{aligned}$$

Conclusion

Summary

- The elliptic quantum group $U_{q,p}(X_l^{(1)})$ is defined as an elliptic and dynamical analogue of the Drinfeld realization of the quantum affine algebra $U_q(X_l^{(1)})$.
- H-Hopf algebroid is a bimodule analogue of graded Hopf algebra, admitting two tensor product $\widetilde{\otimes}$ and $\widehat{\otimes}$ corresponding to Δ and μ respectively.
- An elliptic quantum group possesses an H-Hopf algebroid structure. The $\mathbb{F}$-action is given by the multiplication of the meromorphic functions of dynamical paramaters.

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