Dg symplectic geometry and (higher) Poisson vertex algebras

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Smooth functions $C^\infty(M)$ over a smooth manifold M form a commutative algebra.
In differential-graded geometry, we replace the commutative algebra by a differential graded-commutative algebra.

Example (Shifted tangent bundle)

$T[1]M$:shifted tangent bundle (There $[k]$ means degree shifting the fibres of the vector bundle.)

$$C^\infty(T[1]M) = \Omega^\bullet(M)$$

is the de-Rham complex of M. The de-Rham differential d is a comological vector field on $T[1]M$.

Dg symplectic geometry is symplectic geoemtry in the dg setting.

A dg symplectic manifold $(\mathcal{M}, \omega, S)$
$\mathcal{M}$: a locally ringed space $(M, C^\infty(\mathcal{M}))$ which is locally isomorphic to $(U, C^\infty(U) \otimes \mathrm{Sym} V^*)$, where $U \subset \mathbb{R}^n$ is open, and V is a finite-dimensional graded vector space.
ω:A graded symplectic form of degree k on $\mathcal{M}$ (closed, non-degenerate 2-form)
We can define a degree $-n$ Poisson bracket $\{-,-\}$ on $C^\infty(\mathcal{M})$ via

$$\{f, g\} := (-1)^{|f|+1} X_f(g)$$

where X_f is the unique graded vector field that satisfies $\iota_{X_f}\omega = df$(a Hamiltonian vector field of f).
S:A function (called Hamiltonian) such that $|S| = n + 1$ and $\{S, S\} = 0$. (classical master equation)
We can define a differential on $C^\infty(\mathcal{M})$ by

$$Q = \{S, -\}.$$

We can check $|Q| = 1$ and $Q^2 = 0$.

Important properties:

- It is the mathematical counterpart of BV(BFV) formalism in theoretical physics.
- Some important structures (Poisson manifolds, Courant algebroids, etc.) can be encoded in dg geometric structure.
- Using dg geometric techniques, We can make TFTs and currents in a uniform way.

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BV formalism: a cohomological formalism for treating general gauge theories;

BV formalism	dg symplectic geometry
the space of BV fields	graded manifold
BV antibracket	Poisson bracket induced by a graded symplectic form
BRST operator	cohomological vector field
BV action	a function S satisfying $\{S, S\} = 0$

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Definition

A Poisson bracket on a manifold M is a bilinear operation $C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$, $(f, g) \mapsto \{f, g\}$ satisfying
(1)skew-symmetry $\{f, g\} = -\{g, f\}$
(2)Jacobi-identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
(3)Leibniz identity $\{f, gh\} = g\{f, h\} + \{f, g\}h.$
$(M, \{,\})$ is a Poisson manifold.

Poisson geometry is an active field of research, and has connections with a number of areas.

$k = 1$. Every graded symplectic manifold of degree 1 is canonically isomorphic to the graded cotangent bundle $T^*[1]M$ of the base manifold M. We denote the coordinates of degree 0 by x^i, and the coordinates in degree 1 by p_i.
The Hamiltonian S has degree 2, thus locally it must be of the form,

$$S = \frac{1}{2} \sum_{i,j=1}^{n} \pi^{ij}(x) p_i p_j.$$

Hence, locally S corresponds to a bivector field $\Pi = \pi^{ij}(x) \partial_i \wedge \partial_j$ and $\{S, S\} = 0$ implies that S corresponds to a Poisson bivector field.

$(\mathcal{M}, \omega, S)$: degree 1 dg symplectic manifold
Denote $C_i(C^\infty(\mathcal{M})) = \{f \in C^\infty(\mathcal{M}) : |f| = i\}$. Then

$$C_0(C^\infty(\mathcal{M})) \simeq C^\infty(M).$$

For $f, g \in C_0(C^\infty(\mathcal{M}))$,

$$\begin{aligned}\{\{f, S\}, g\} &= \{\sum_{i,j=1}^{n} \frac{\partial f}{\partial x^i} \pi^{ij} p_j, g\} \\ &= \sum_{i,j=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \pi^{ij},\end{aligned}$$

which is a Poisson manifold structure.
Hence, there is a one-to-one correspondence between isomorphism class of dg symplectic manifolds of degree 1 and isomorphism class of Poisson manifolds.

Definition (Liu-Weinstein-Xu 95)

A Courant algebroid is a vector bundle E over a smooth manifold M, with a non-degenerate symmetric bilinear form $\langle , \rangle$, and a bilinear bracket $*$ on $\Gamma(E)$. The form and the bracket must be compatiable, in the meaning defined below, with the vector fields on M. We must have a smooth bundle map, the anchor

$$\pi : E \to TM.$$

These structure satisfy the following five axioms, for all $A, B, C \in \Gamma(E)$ and $f \in C^\infty(M)$.

- Axiom.1 : $\pi(A * B) = [\pi(A), \pi(B)]$ (The bracket of the right hand side is the Lie bracket of vector fields).
- Axiom.2 : $A * (B * C) = (A * B) * C + B * (A * C)$.
- Axiom.3 : $A * (fB) = (\pi(A)f)B + f(A * B)$.
- Axiom.4 : $\langle A, B * C + C * B \rangle = \pi(A)\langle B, C \rangle$.
- Axiom.5 : $\pi(A)\langle B, C \rangle = \langle A * B, C \rangle + \langle B, A * C \rangle$.

Example (Courant 90)

Dorfman bracket on $TM \oplus T^*M$

$$\langle (v_1, \alpha_1), (v_2, \alpha_2) \rangle = \iota_{v_1}\alpha_2 + \iota_{v_1}\alpha_2,$$

$$[(v_1, \alpha_1), (v_2, \alpha_2)] = ([v_1, v_2], L_{v_1}\alpha_2 + \frac{1}{2}d(\iota_{v_2}\alpha_1 - \iota_{v_1}\alpha_2) - L_{v_2}\alpha_1).$$

Courant algebroids play an important role in some areas of mathematics and physics, for example, generalized geomtries, T-dualities, topological sigma models, supergravity, and double field theories.

$(\mathcal{M}, \omega, S)$: degree 2 dg symplectic manifold
Denote $C_i(C^\infty(\mathcal{M})) = \{f \in C^\infty(\mathcal{M}) : |f| = i\}$. Then

$$C_0(C^\infty(\mathcal{M})) \simeq C^\infty(M), C_1(C^\infty(\mathcal{M})) \simeq \Gamma(E).$$

For $f \in C_0(C^\infty(\mathcal{M}))$ and $A, B \in C_1(C^\infty(\mathcal{M}))$, we define

$$\begin{aligned} \{A, B\} &=: \langle A, B\rangle \\ \{\{A, S\}, B\} &=: A * B \\ \{\{A, S\}, f\} &=: \pi(A) f = \partial(f) A = \{\{S, f\}, A\}. \end{aligned}$$

This computation gives a dg symplectic manifold of degree 2 the structure of a Courant algebroid.

Theorem (Roytenberg 02)

Dg symplectic manifolds of degree 2 are in 1-1 correspondence with Courant algebroids.

Important properties:

- It is the mathematical counterpart of BRST and BV(BFV) formalism in theoretical physics.
- Some important structures (Poisson manifolds, Courant algebroids, etc.) can be encoded in dg geometric structure.
- Using dg geometric techniques, We can make TFTs and currents in a uniform way.

AKSZ sigma models: $(n+1)$-dimensional Topological field theories whose target datum are degree n dg symplectic manifolds.
Examples:

- Poisson sigma models (2-dimensional TFTs whose targets are Poisson manifolds)
- Chern-Simons sigma models (3-dimensional TFTs whose targets are Lie algebras)

BFV current algebras: $(n-1)$-dimensional current algebras whose target datum are degree n dg symplectic manifolds.
$n=2$:They have a relation with Poisson vertex algebras.

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Current algebras: Poisson algebras consisting of functions on mapping spaces.

Example (Alekseev-Strobl 04)

Current algebras on $C^\infty(T^*LM)$ $(LM = \mathrm{Map}(S^1, M))$

$$J_{(v,\alpha)}(\sigma) = v^i(x(\sigma))p_i(\sigma) + \alpha_i(x(\sigma))\partial_\sigma x^i(\sigma).$$

$$\{J_{(v,\alpha)}(\sigma), J_{(u,\beta)}(\sigma')\} = J_{[(v,\alpha),(u,\beta)]}(\sigma)\delta(\sigma - \sigma') + \langle (v, \alpha), (u, \beta)\rangle(\sigma)\delta'(\sigma - \sigma'),$$

u, v is a vector field on M, α, β is a 1-form on M, $[(v, \alpha), (u, \beta)] = ([v, u], L_v\beta - \iota_u d\alpha)$ is the Dorfman bracket on the generalized tangent bundle $TM \oplus T^*M$ and $\langle (v, \alpha), (u, \beta)\rangle = \iota_u\alpha + \iota_v\beta$.

Alekseev-Strobl current algebras can be described in the language of dg symplectic geometry.
Poisson algebras on the mapping space
$n-1$ dimensional manifolds→degree n dg symplectic manifolds
were constructed. They are called BFV(Batalin-Fradkin-Vilkovisky) current algebras.

BFV current algebras (Ikeda-Koizumi 11, Ikeda-Xu 13, Arvanitakis 21)

$$J_\epsilon(\alpha)(\phi) = \int_{T[1]\Sigma_{n-1}} \epsilon \cdot \phi^*(a)(\sigma,\theta) d^{n-1}\sigma d^{n-1}\theta,$$

$$\{J_{\epsilon_1}(\alpha), J_{\epsilon_2}(\beta)\} = \int_{T[1]\Sigma_{n-1}} \epsilon_1\epsilon_2\{\{\alpha,\Theta\},\beta\}(\sigma,\theta)d^{n-1}\sigma d^{n-1}\theta + \int_{T[1]\Sigma_{n-1}} (d\epsilon_1)\epsilon_2\{\alpha,\beta\}(\sigma,\theta)d^{n-1}\sigma d^{n-1}\theta.$$

They can be seen as a higher generalization of Alekseev-Strobl current algebras.

We consider $C^\infty(Map(T[1]\Sigma_{n-1}, \mathcal{M}))$, where $(T[1]\Sigma_{n-1}, D)$ is the shifted tangent space of Σ_{n-1} and $(\mathcal{M}, \omega, Q = \{\Theta, -\})$ is a degree n dg symplectic manifold.
BFV currents are on $C^\infty(Map(T[1]\Sigma_{n-1}, \mathcal{M}))/I_{\tilde{D}+\tilde{Q}}$, where $\tilde{D}$ and $\tilde{Q}$ is a differential on $Map(T[1]\Sigma_{n-1}, \mathcal{M})$ induced by D and Q.
For $a \in C^\infty(\mathcal{M})$ and $\epsilon \in C^\infty(T[1]\Sigma_{n-1})$, We define $J_\epsilon(a) \in C^\infty(Map(T[1]\Sigma_{n-1}, \mathcal{M}))$ by

$$J_\epsilon(a)(\phi) = \int_{T[1]\Sigma_{n-1}} \epsilon \cdot \phi^*(a)(\sigma, \theta) d^{n-1}\sigma d^{n-1}\theta,$$

where $\epsilon \in C^\infty(T[1]\Sigma_{n-1})$ are test functions on $T[1]\Sigma_{n-1}$, σ, θ are coordinates on $T[1]\Sigma_{n-1}$ of degree 0 and 1, $\phi \in Map(T[1]\Sigma_{n-1}, \mathcal{M})$ and $\phi^*(a)$ is the pullback of a.

Then the Poisson bracket is

$$\begin{aligned}
&\{J_{\epsilon_1}(a), J_{\epsilon_2}(b)\}(\phi) \\
&= \int_{T[1]\Sigma_{n-1}} \epsilon_1\epsilon_2 \cdot \phi^*(\{\{a, \Theta\}, b\})(\sigma, \theta) d^{n-1}\sigma d^{n-1}\theta \\
&+ \int_{T[1]\Sigma_{n-1}} (D\epsilon_1)\epsilon_2 \cdot \phi^*(\{a, b\})(\sigma, \theta) d^{n-1}\sigma d^{n-1}\theta,
\end{aligned}$$

where $\epsilon_1, \epsilon_2 \in C^\infty(T[1]\Sigma_{n-1})$ are test functions on $T[1]\Sigma_{n-1}$, σ, θ are coordinates on $T[1]\Sigma_{n-1}$ of degree 0 and 1,

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$\langle (v,\alpha),(u,\beta)\rangle = \iota_u\alpha + \iota_v\beta$.

an algebraic generalization of
Courant algebroids: Courant-Dorfman algebras
Alekseev-Strobl currents: Poisson vertex algebras
Ekstrand gave weak Couarnt-Dorfman algebras from Lie conformal algebras and showed that the graded Poisson vertex algebras generated by elements of degree 0 and 1 are in one-to-one correspondence with the Courant-Dorfman algebras.

Definition (Roytenberg 09)

A Courant-Dorfman algebra consists of the following data:

- a commutative algebra R,
- an R-module E,
- a symmetric bilinear form $\langle , \rangle : E \otimes E \to R$,
- a derivation $\partial : R \to E$,
- a Dorfman bracket $[,] : E \otimes E \to E$,

which satisfies the following conditions;

$$[e_1, fe_2] = f[e_1, e_2] + \langle e_1, \partial f \rangle e_2, \langle e_1, \partial \langle e_2, e_3 \rangle \rangle = \langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle,$$

$$[e_1, e_2] + [e_2, e_1] = \partial \langle e_1, e_2 \rangle, [e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]],$$

$$[\partial f, e] = 0, \langle \partial f, \partial g \rangle = 0,$$

where $f, g \in R$ and $e_1, e_2, e_3 \in E$.

Definition

A Lie conformal algebra is a $\mathbb{C}[\partial]$-module W (i.e.∂ acts on elements of W) with a λ-bracket $\{_\lambda\} : W \otimes W \to W[\lambda]$, $\{a_\lambda b\} = \sum_{j\in\mathbb{Z}_+} \lambda^j a_{(j)} b$ which satisfies the conditions.

Sesquilinearity :$\{\partial a_\lambda b\} - \lambda\{a_\lambda b\},\ \{a_\lambda \partial b\} = (\partial + \lambda)\{a_\lambda b\}$

Skew-symmetry :$\{a_\lambda b\} = -\{b_{-\lambda-\partial} a\}$

Jacobi-identity :$\{a_\lambda\{b_\mu c\}\} = \{\{a_\lambda b\}_{\mu+\lambda} c\} + \{b_\mu\{a_\lambda c\}\}$

A Poisson vertex algebra is a commutative algebra W with a derivation ∂(i.e.$\partial(ab) = (\partial a)b + a(\partial b)$) and λ-bracket $\{_\lambda\} : W \otimes W \to W[\lambda]$ such that W is a Lie conformal algebra and satisfies the Leibniz rule.

Leibniz rule :$\{a_\lambda b \cdot c\} = \{a_\lambda b\} \cdot c + b \cdot \{a_\lambda c\}$

Theorem (Ekstrand,11)

The Poisson vertex algebras that are graded and generated by elements of degree 0 and 1 are in a one-to-one correspondence with the Courant-Dorfman algebras.

We can consider a notion of a higher version of the relation between Courant-Dorfman algebras and Poisson vertex algebras.
an algebraic generalization of
functions of degree n dg symplectic manifolds: higher Courant-Dorfman algebras
BFV currents:higher Poisson vertex algebras
With higher Courant-Dorfman algebras and higher Poisson vertex algebras, we may be able to find and unify more general current algebras including the BFV current algebras, and use the techniques of Poisson vertex algebras in the higher setting.

$R = E^0$:a commutative algebra
$E = \oplus_{i=0}^{n-1} E^i$:a graded R-module
$\langle , \rangle : E \otimes E \to R$ a pairing such that $\langle a, b \rangle = 0$ unless $|a| + |b| = n$
Consider the graded-commutative algebra freely generated by E and denote it by $\tilde{\mathcal{E}} = (\mathcal{E}^k)_{k \in \mathbb{Z}}$. We restrict this graded-commutative algebra to the elements of degree $n - 1 \geq k \geq 0$ and denote it by $\mathcal{E} = (\mathcal{E}^k)_{n-1 \geq k \geq 0}$. The pairing $\langle , \rangle$ can be extended to $\mathcal{E}$ by the Leibniz rule

$$\langle a, b \cdot c \rangle = \langle a, b \rangle \cdot c + (-1)^{(|a|-n)|b|} b \cdot \langle a, c \rangle.$$

Definition

$\mathcal{E} = (\mathcal{E}^k)$ is a *higher Courant-Dorfman algebra* if $\mathcal{E}$ has a differential $d : \mathcal{E}^k \to \mathcal{E}^{k+1}$ which satisfies $d^2 = 0$ and $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$ and a bracket $[,] : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ of degree $1 - n$ which satisfies the following condition:

sesquilinearity :

$$\langle da, b\rangle = -(-1)^{|a|-n}[a, b], [da, b] = 0.$$

skew-symmetry :

$$[a, b] + (-1)^{(|a|+1-n)(|b|+1-n)}[b, a] = -(-1)^{|a|}d\langle a, b\rangle,$$

$$\langle a, b\rangle = -(-1)^{(|a|-n)(|b|-n)}\langle b, a\rangle.$$

Jacobi identity :

$$[a, [b, c]] = [[a, b], c] + (-1)^{(|a|+1-n)(|b|+1-n)}[b, [a, c]],$$

$$[a, \langle b, c\rangle] = \langle [a, b], c\rangle + (-1)^{(|a|+1-n)(|b|+1-n)}\langle b, [a, c]\rangle,$$

$$\langle a, \langle b, c\rangle\rangle = \langle\langle a, b\rangle, c\rangle + (-1)^{(|a|-n)(|b|-n)}\langle b, \langle a, c\rangle\rangle.$$

Leibniz rule :

$$[a \cdot b, c] = [a, b] \cdot c + (-1)^{(|a|+1-n)|b|}b \cdot [a, c].$$

Definition

Let $C = (C^n, d)$ a cochain complex. C is a higher Lie conformal algebra of degree n if it endows with a degree $1-n$ map which we call Λ-bracket $[_\Lambda] : C \otimes C \to C[\Lambda]$ defined by

$$a \otimes b \mapsto [a_\Lambda b] = a_{(0)}b + \Lambda a_{(1)}b$$

satisfying the conditions.

Sesquilinearity

$$[da_\Lambda b] = -(-1)^{-n}\Lambda[a_\Lambda b], [a_\Lambda db] = -(-1)^{|a|-n}(d+\Lambda)[a_\Lambda b]$$

Skewsymmetry

$$[a_\Lambda b] = -(-1)^{(|a|+1-n)(|b|+1-n)}[b_{-\Lambda-d}a]$$

Jacobi identity

$$[a_\Lambda[b_\Gamma c]] = [[a_\Lambda b]_{\Lambda+\Gamma}c] + (-1)^{(|a|+1-n)(|b|+1-n)}[b_\Gamma[a_\Lambda c]].$$

Definition

C is a higher Poisson vertex algebra if it is a higher LCA and a differential graded-commutative algebra which satisfies

the Leibniz rule

$$[a_\Lambda bc] = [a_\Lambda b]c + (-1)^{(|a|+1-n)|b|} b[a_\Lambda c].$$

Theorem (H,23)

There is a one-to-one correspondence between (dg) higher Poisson vertex algebras generated by elements of degree $0 \leq i \leq n-1$ and extended higher Courant-Dorfman algebras.

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A symplectic structure or a Poisson structure is defined on a commutative algebra.
Non-commutative analog: a bi-symplectic structure, double Poisson algebra

Definition (Van den Bergh, 08)

An associative unital algebra $\mathbb{A}$ is a double Poisson algebra if it is equipped with

$$\{\{-,-\}\}:\mathbb{A}\otimes\mathbb{A}\to\mathbb{A}\otimes\mathbb{A},\ \ a\otimes b\mapsto\{\{a,b\}\}$$

skew-symmetry :

$$\{\{a,b\}\}=-\{\{a,b\}\}^{\sigma},$$

Jacobi identity :

$$\{\{a,\{\{b,c\}\}\}\}_L-\{\{b,\{\{a,c\}\}\}\}_R-\{\{\{\{a,b\}\},c\}\}_L=0$$

$$\{\{a,b'\otimes b''\}\}_L:=\{\{a,b'\}\}\otimes b'',\ \ \{\{a,b'\otimes b''\}\}_R:=b'\otimes\{\{a,b''\}\}.$$

Leibniz rule :

$$\{\{a,bc\}\}=\{\{a,b\}\}c+b\{\{a,c\}\}.$$

Dg bisymplectic algebras (bisymplectic analog of dg symplectic geometry) and double Poisson vertex algebras (double Poisson analog of PVA) were studied and noncommutative analog of the one-to-one correspondence was shown.

Theorem (Alvarez-Consul,Fernandez,Heluani 21)

There exists a one-to-one correspondence between graded double Poisson vertex algebras of degree -1, freely generated in weights 0 and 1, and double Courant-Dorfman algebras.

We can expect the higher generalization of the correspondence can be constructed in terms of dg bisymplectic algebras, like the ordinary PVA case, but the physical meaning is ambiguous.

Thank you.