Tight Cramér-Rao type bounds for multiparameter quantum metrology through conic programming

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What is quantum metrology?

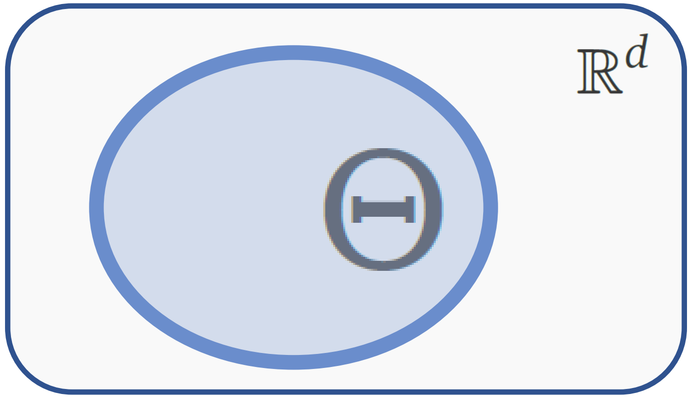
- Given a **quantum model**, which is set of quantum states

$$\mathcal{M} := \{\rho_\theta \,|\, \theta \in \Theta\}$$

↑

Probe state, must be differentiable

$\partial \rho_\theta / \partial \theta_i$ to be linearly independent



$\mathcal{H}$	the Hilbert space for the quantum probe state
n	the number of dimensions $\mathcal{H}$ has

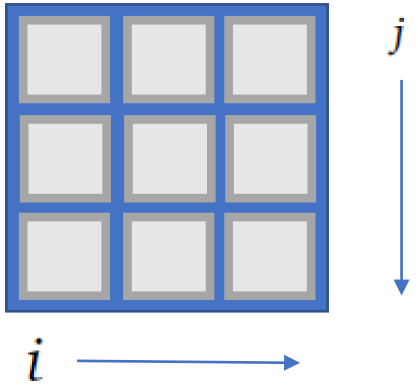
- We perform a measurement on the probe state.
- The probability distribution we obtain depends on θ.
- Estimate θ with the minimum 'error'.

d	the number of parameters to be estimated
$\theta = (\theta^1, \ldots, \theta^d)$	the parameters' true value
$\Theta \subseteq \mathbb{R}^d$	set of all possible parameter vectors
$\mathcal{M}$	model $\{\rho_\theta : \theta \in \Theta\}$

Precision bound for $\mathcal{M} := \{\rho_\theta \mid \theta \in \Theta\}$

mean-square error (MSE) matrix

$$V_\theta[\hat{\Pi}] = \left[\sum_{x\in\mathcal{X}} \mathrm{Tr}[\rho_\theta \Pi_x](\hat{\theta}^i(x)-\theta^i)(\hat{\theta}^j(x)-\theta^j)\right]$$
$$= \left[E_\theta[(\hat{\theta}^i(x)-\theta^i)(\hat{\theta}^j(x)-\theta^j)|\Pi]\right].$$



an estimator $\hat{\theta}$ POVM $\Pi = \{\Pi_x\}_{x\in\mathcal{X}}$

$$\hat{\Pi} = (\Pi, \hat{\theta})$$

weight matrix

minimize $\mathrm{Tr}[G V_\theta[\hat{\Pi}]]$

Precision bound:

$$\text{minimize} \quad \mathrm{Tr}[G V_\theta[\hat{\Pi}]]$$

$\hat{\Pi}$:l.u.at θ

$$E_\theta[\hat{\theta}^i(X)|\Pi] = \sum_{x \in \mathcal{X}} \hat{\theta}^i(x) \mathrm{Tr}[\rho_\theta \Pi_x] = \theta^i,$$

$$\frac{\partial}{\partial \theta^j} E_\theta[\hat{\theta}^i(X)|\Pi] = \sum_{x \in \mathcal{X}} \hat{\theta}^i(x) \mathrm{Tr}\left[\frac{\partial}{\partial \theta^j} \rho_\theta \Pi_x\right] = \delta_i^j.$$

fundamental precision limit

$$C_\theta[G] := \min_{\hat{\Pi} \text{ :l.u.at } \theta} \mathrm{Tr}\left[G V_\theta[\hat{\Pi}]\right]$$

Lower bounds on $C_\theta[G]$

SLD Fisher information matrix

SLD CR bound $\quad C^S[G] := \mathrm{Tr}[GJ^{-1}]$

$$J_{i,j} := \frac{1}{2}\mathrm{Tr}\left[L_i\left(L_j\rho + \rho L_j\right)\right] \qquad D_i = \frac{1}{2}(L_i\rho + \rho L_i)$$

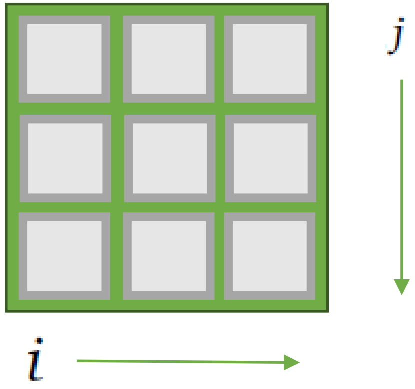
$D_j = \frac{\partial}{\partial \theta^j}\rho_\theta$ $\quad j$th partial derivative of ρ_θ

Holevo-Nagaoka (HN) bound

$$\vec{Z} = (Z^1, \ldots, Z^d) \qquad \mathrm{Tr}[D_j Z^i] = \delta^j_i \text{ for } i, j = 1, \ldots, d.$$

$$V[\hat{\Pi}] \geq \mathcal{Z}(\vec{Z})$$

$$\mathcal{Z}(\vec{Z}) \longleftarrow \mathrm{Tr}\rho Z^i Z^j$$



$$\mathrm{Tr} G V_\theta[\hat{\Pi}] \geq \mathrm{Tr} G \mathrm{Re}\, \mathcal{Z}(\vec{Z}) + \mathrm{Tr}|G^{\frac{1}{2}} \mathrm{Im}\, \mathcal{Z}(\vec{Z}) G^{\frac{1}{2}}|$$

$|X|$ is defined as $\sqrt{X^\dagger X}$

$$C^{HN}[G] := \min_{\vec{Z}=(Z^1,\ldots,Z^d)} \mathrm{Tr}[G \mathrm{Re}\, \mathcal{Z}(\vec{Z})] + \mathrm{Tr}[|G^{\frac{1}{2}} \mathrm{Im}\, \mathcal{Z}(\vec{Z}) G^{\frac{1}{2}}|]$$

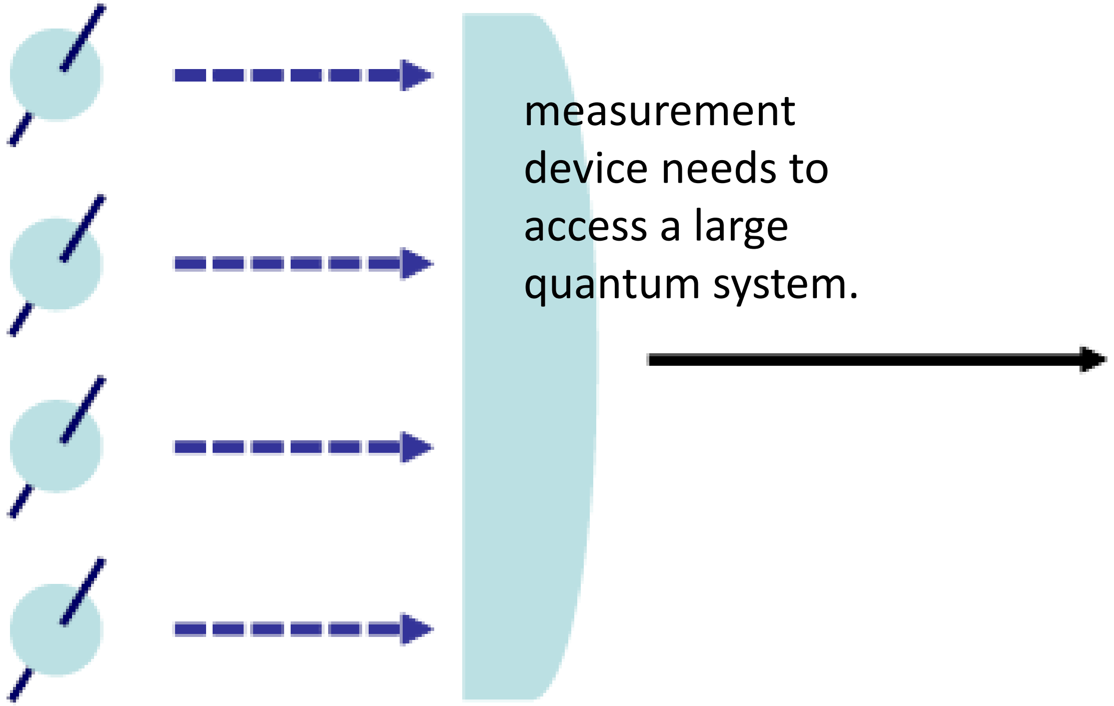
Nagaoka [4] proved this using inequalities from Holevo [3].

[3] **Holevo**, *Probabilistic and statistical aspects of quantum theory* (Edizioni della Normale, 2011).
[4] **Nagaoka**, A new approach to Cramér-Rao bounds for quantum state estimation, IEICE Tech Report IT 89-42, 9 (1989)

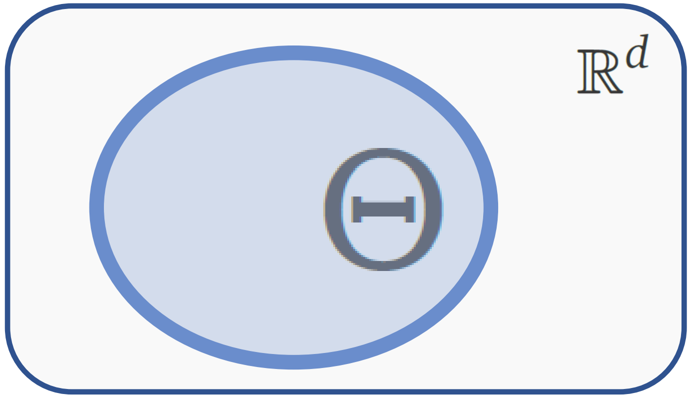
Holevo-Nagaoka (HN) bound

Correlated measurement

Estimate



$$\hat{\theta}$$



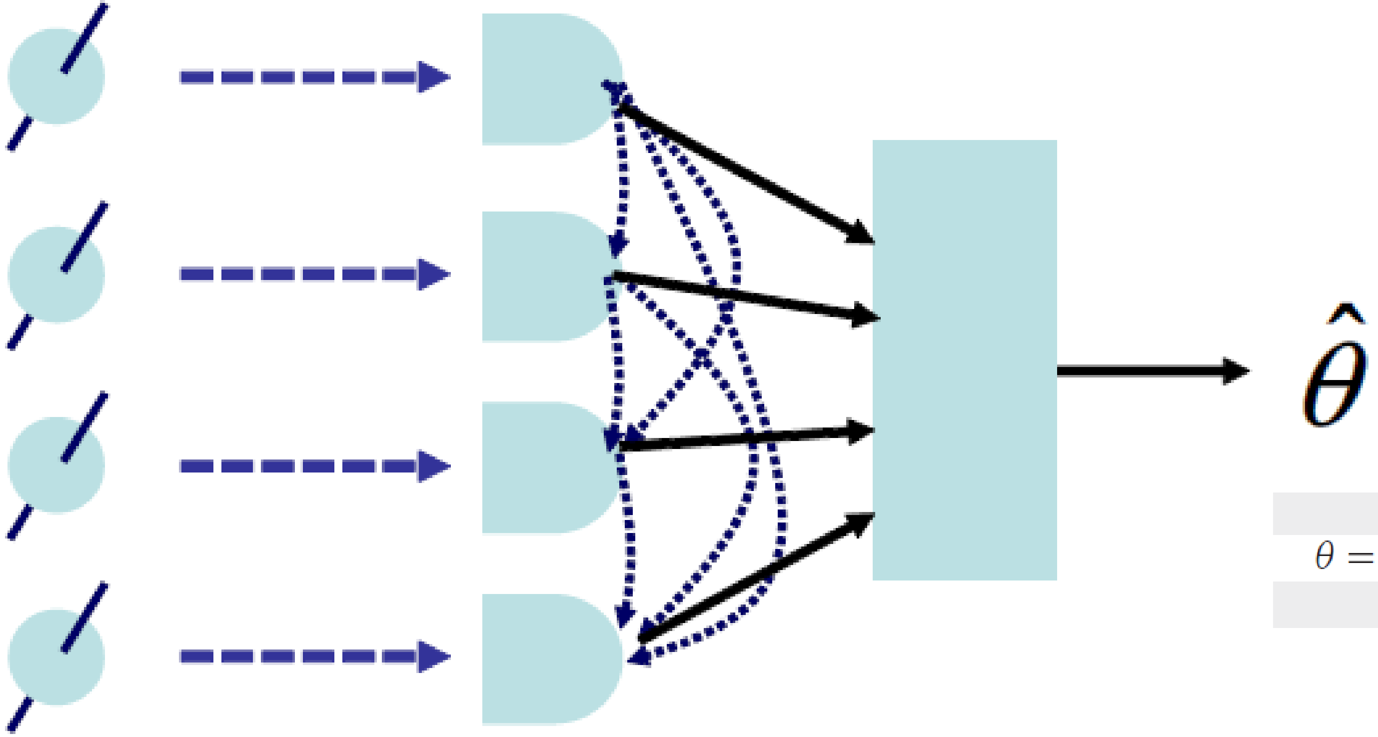
d	the number of parameters to be estimated
$\theta = (\theta^1, \dots, \theta^d)$	the parameters' true value
$\Theta \subseteq \mathbb{R}^d$	set of all possible parameter vectors
$\mathcal{M}$	model $\{\rho_\theta : \theta \in \Theta\}$

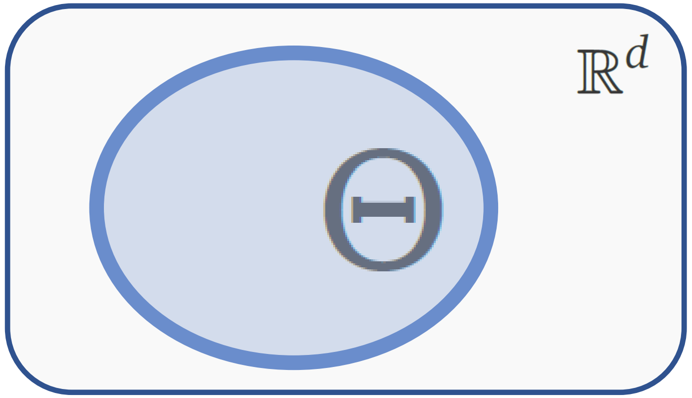
identical copies of the probe states.

$\hat{\theta} = (\hat{\theta}^1, \dots, \hat{\theta}^d)$ an estimator of θ

Uncorrelated measurement

Estimate





classical feedback is allowed to improve the measurement

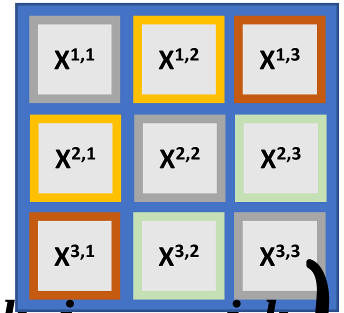
d	the number of parameters to be estimated
$\theta = (\theta^1, \ldots, \theta^d)$	the parameters' true value
$\Theta \subseteq \mathbb{R}^d$	set of all possible parameter vectors
$\mathcal{M}$	model $\{\rho_\theta : \theta \in \Theta\}$

- The Nagaoka bound [4, 10], which is given only in the case with $d=2$, and is tighter than the HN bound.

$$C_\theta^N[G] := \min_{\vec{Z}=(Z^1,Z^2)} G_{1,1}\mathrm{Tr}[Z^1\rho Z^1] + G_{2,2}\mathrm{Tr}[Z^2\rho Z^2] + G_{1,2}\mathrm{Tr}[\rho(Z^1Z^2+Z^2Z^1)] + 2\sqrt{\det G}\,\mathrm{Tr}[|\rho^{1/2}[Z^1,Z^2]\rho^{1/2}|],$$

$$\mathrm{Tr}[D_j Z^i] = \delta_i^j \text{ for } i,j=1,\ldots,d.$$

- The Nagaoka-Hayashi (NH) bound [12], which is tighter than the HN bound.



$$C^{NH}[G] := \min_{\vec{Z}} \min_{X'\in\mathcal{B}'} \{\mathrm{Tr}[(G\otimes\rho)X'] | X' \geq \Pi(\vec{Z})\}, \quad (22)$$

$$\mathcal{B}' := \left\{ \sum_{j,k=1}^{d} |k\rangle\langle j| \otimes X^{k,j} \,\middle|\, X^{k,j} \in \mathcal{B}_{sa}(\mathcal{H}), X^{k,j} = X^{j,k} \right\}$$

$$\Pi(\vec{Z}) := (Z^i Z^j)_{i,j}$$

$\mathcal{H}$	the Hilbert space for the quantum probe state
n	the number of dimensions $\mathcal{H}$ has

[4] **Nagaoka**, A new approach to Cramér-Rao bounds for quantum state estimation, IEICE Tech Report **IT 89-42**, 9 (1989)
[10] **Nagaoka**, A generalization of the simultaneous diagonalization of hermitian matrices and its relation to quantum estimation theory, in *Asymptotic Theory Of Quantum Statistical Inference: Selected Papers*, edited by M. Hayashi (World Scientific, 2005) pp. 133–149.
[12] **Conlon, Suzuki, Lam, Assad**, Efficient computation of the NH bound for multiparameter estimation with separable measurements, npj Quantum Information **7**, 1 (2021).

tight CR bound

$$C_\theta[G] := \min_{\hat{\Pi}:\text{l.u.at }\theta} \mathrm{Tr}\left[G V_\theta[\hat{\Pi}]\right]$$

[13] **Hayashi**, A linear programming approach to attainable cramér-rao type bounds and randomness condition (1997), arXiv:quant-ph/9704044

[37] **Hayashi**, A linear programming approach to attainable Cramer-Rao type bound, in *Quantum Communication, Computing, and Measurement*, edited by O. Hirota, A. S. Holevo,and C. M. Caves (Plenum, New York, 1997)

Proposition 1 ([13, Theorem 6]). $C[G] = S(D0)$.

$$S(D0) := \max_{a,S} \sum_i a_i^i + \mathrm{Tr} S \quad \text{subject to} \quad (x^T G x)\rho - \sum_{i,j} a_i^j x^i D_j - S \geq 0$$

Outstanding questions:

(1): How to efficiently determine the **tight bound**?

(2): How to determine the optimal uncorrelated measurement strategy for multiparameter quantum metrology that saturates the **tight bound**?

(3): What is the relationship between the **SLD bound**, the **HN bound**, the **NH bound**, and the **tight bound**?

(4): Is there a gap between the **tight bound** and the **NH bound**?

Main results

- (1) & (2): We efficiently compute optimal uncorrelated measurement strategies.

- (3): We unify the theory of the SLD, HN, NH and tight bounds under a common umbrella of **conic programming**.

- (4): Using our algorithm, we numerically demonstrate that the tight
- bound can be strictly tighter than the NH bound.

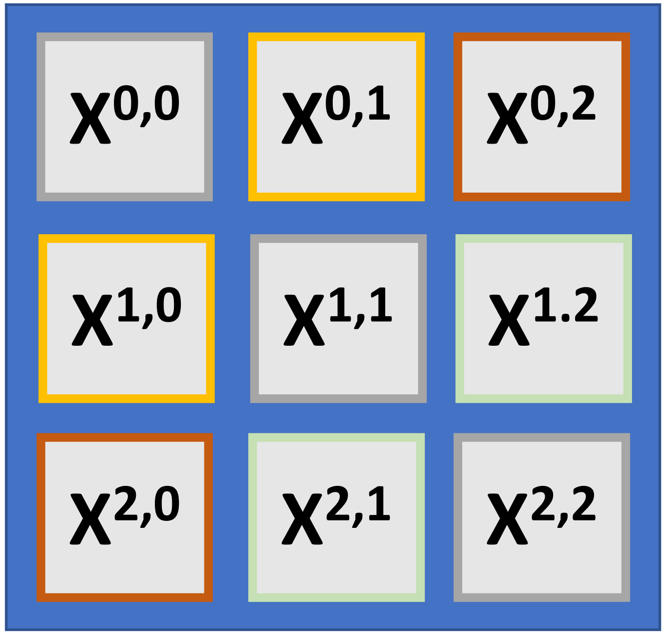
Conic programming

$\mathcal{R}$	real vector space spanned by $\|0\rangle, \|1\rangle, \ldots, \|d\rangle$
$\mathcal{R}_C$	complex vector space spanned by $\|0\rangle, \ldots, \|d\rangle$

= $\mathcal{H}$ = $\mathcal{R}_C$ = $\mathcal{R}$

Block matrix **X** is optimization variable

X



Cones:

$$\mathcal{S}_{SEP} := conv(\mathcal{M}_{rs,+}(\mathcal{R}) \otimes \mathcal{B}_{sa,+}(\mathcal{H}))$$

$$\mathcal{S}_P := \{X \in \mathcal{B} | \langle v|X|v\rangle \geq 0 \text{ for } v \in \mathcal{R}_C \otimes \mathcal{H}\}$$

$$\underbrace{\mathcal{S}_{SEP}}_{P1} \subset \underbrace{\mathcal{S}_P}_{P2} \subset \underbrace{\mathcal{S}(\mathcal{R}_C \otimes \mathcal{H})_{PPT} \cap \mathcal{B}''}_{P3} \subset \underbrace{\mathcal{S}(\mathcal{R}_C \otimes \mathcal{H})_P \cap \mathcal{B}''}_{P4}.$$

Matrix spaces:

$$\mathcal{B}' := \left\{ \sum_{j=1}^{d} \sum_{k=1}^{d} \middle| |k\rangle\langle j| \otimes X^{k,j} | X^{k,j} \in \mathcal{B}_{sa}(\mathcal{H}), X^{k,j} = X^{j,k} \right\}$$

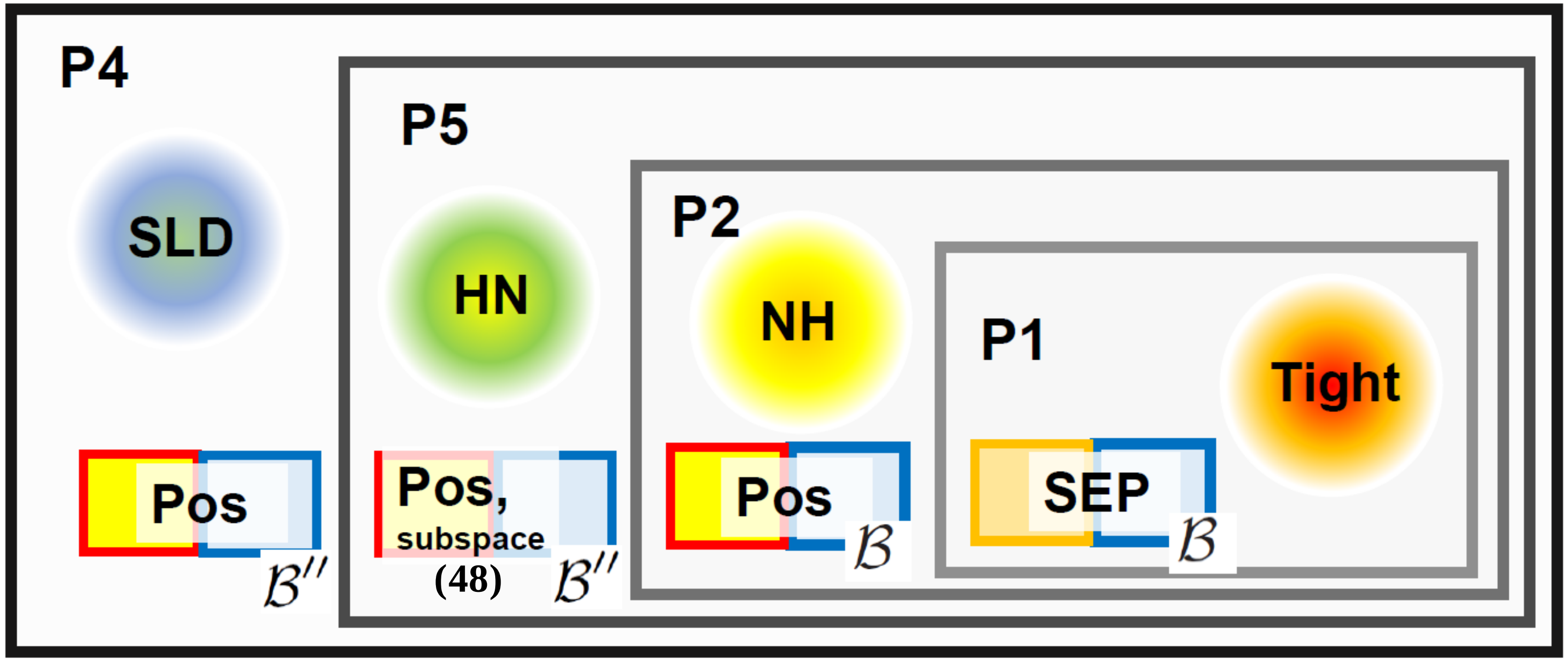
$$\mathcal{B} := \left\{ \sum_{j=0}^{d} \sum_{k=0}^{d} |k\rangle\langle j| \otimes X_{k,j} \middle| X_{k,j} \in \mathcal{B}_{sa}(\mathcal{H}), X_{k,j} = X_{j,k} \right\}.$$

$$\mathcal{B}'' := \left\{ \sum_{j=0}^{d} \sum_{k=0}^{d} |k\rangle\langle j| \otimes X_{k,j} \middle| X_{k,0} \in \mathcal{B}_{sa}(\mathcal{H}), X_{k,j} = (X_{j,k})^{\dagger} \right\}$$

Conic programming

$$\mathcal{S}_{SEP} \subset \mathcal{S}_{P} \subset \mathcal{S}(\mathcal{R}_C \otimes \mathcal{H})_{PPT} \cap \mathcal{B}'' \subset \mathcal{S}(\mathcal{R}_C \otimes \mathcal{H})_{P} \cap \mathcal{B}''.$$

P1 P2 P3 P4



Theorem
$C^{NH}[G] = S(P2)$
$C^{S}[G] = S(P4)$

Theorem
$C^{HN}[G] = S(P5)$

$$S(P1) := \min_{X \in \mathcal{S}_{SEP}} \{\mathrm{Tr}[(G \otimes \rho)X] \,|\, (36), (37) \text{ hold.}\}$$

Theorem $C[G] = S(P1)$

$$S(P5) := \min_{X \in \mathcal{S}(\mathcal{R}_C \otimes \mathcal{H})_P \cap \mathcal{B}''} \{\mathrm{Tr}[(G \otimes \rho)X] \,|\, (36), (37), (48) \text{ hold.}\}$$

$$\mathrm{Tr}_{\mathcal{R}}[(|0\rangle\langle 0| \otimes I_{\mathcal{H}})X(\Pi, \hat{\theta})] = I_{\mathcal{H}}. \qquad (36)$$

$$\mathrm{Tr}\left[\left(\frac{1}{2}(|0\rangle\langle i| + |i\rangle\langle 0|) \otimes D_j\right)X(\Pi, \hat{\theta})\right] = \delta_{i,j}. \qquad (37)$$

$$\mathrm{Tr}[X((|j\rangle\langle i| - |i\rangle\langle j|) \otimes \rho)] = 0 \qquad (48)$$

Proposition 1 ([13, Theorem 6]). $C[G] = S(D0)$.

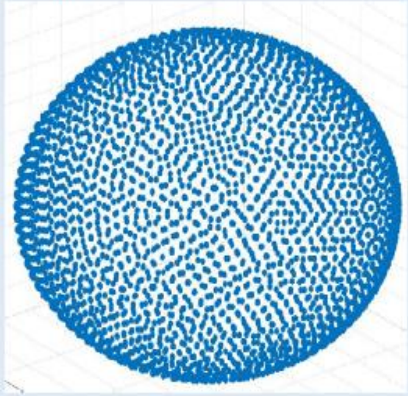
$$S(D0) := \max_{a,S} \sum_i a_i^i + \mathrm{Tr} S \quad \text{subject to} \quad (x^T G x)\rho - \sum_{i,j} a_i^j x^i D_j - S \geq 0$$

Has to hold for all x in $\mathbb{R}^d$

Idea: Consider $\mathcal{W}_R$, a finite subset of $\mathbb{R}^{d+1}$, with only norm 1 vectors.

V: constructing W_R

W_R : finite subset of the hypersphere



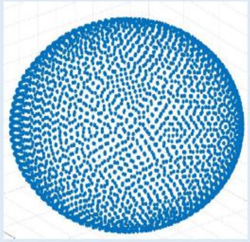
δ is the covering radius of the hypersphere

$\delta = O(\sqrt{d}/|W_R|^{2/d})$

$$\mathcal{W}_R := \{|w_s\rangle\}_{s=1}^{m}$$

V: constructing W_R

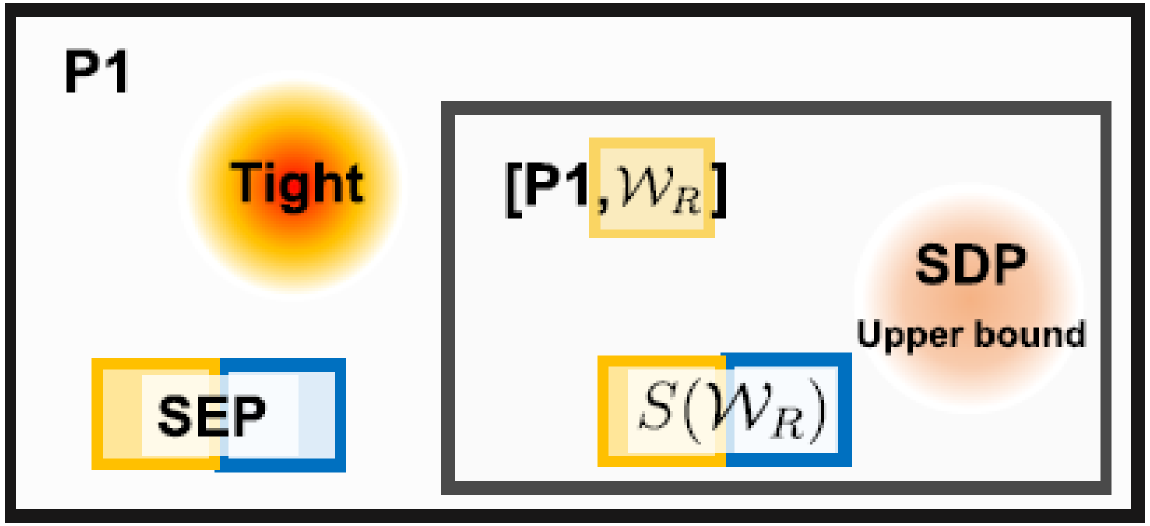
W_R : finite subset of the hypersphere



δ is the covering radius of the hypersphere

$\delta = O(\sqrt{d}/|W_R|^{2/d})$

$$\mathcal{S}(\mathcal{W}_R) := \{\textstyle\sum_{s=1}^{m} |w_s\rangle\langle w_s| \otimes X_s | X_s \in \mathcal{T}_{sa,+}(\mathcal{H})\} \subset \mathcal{S}_{SEP}$$



$$S[P1, \mathcal{W}_R] \geq S(P1)$$

complexity of $S[P1, \mathcal{W}_R]$

$$= O(m(n^6 + d^2 n^3 + d^4 n^2))$$

$$S(P1) := \min_{X \in \mathcal{S}_{SEP}} \left\{ \mathrm{Tr}\left[(G \otimes \rho) X\right] \middle| (36), (37) \text{ hold.} \right\}$$

$$\mathrm{Tr}_{\mathcal{R}}\left[(|0\rangle\langle 0| \otimes I_{\mathcal{H}}) X(\Pi, \hat{\theta})\right] = I_{\mathcal{H}}. \qquad (36)$$

$$\mathrm{Tr}\left[\left(\frac{1}{2}(|0\rangle\langle i| + |i\rangle\langle 0|) \otimes D_j\right) X(\Pi, \hat{\theta})\right] = \delta_{i,j}. \qquad (37)$$

$$S[P1, \mathcal{W}_R] := \min_{X \in \mathcal{S}(\mathcal{W}_R)} \left\{ \mathrm{Tr}\left[(G \otimes \rho) X\right] \middle| (36), (37) \text{ hold.} \right\}$$

$$= \min_{X = (X_s) \in \mathcal{T}(\mathcal{W}_R)} \left\{ \sum_{s=1}^{m} \langle w_s | G | w_s \rangle \mathrm{Tr} X_s \rho \middle| C[P1, \mathcal{W}_R] \text{ holds.} \right\}$$

$$\textstyle\sum_{s=1}^{m} |s\rangle\langle s| \otimes X_s$$

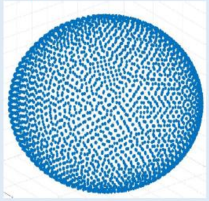
$$\sum_{s=1}^{m} \langle w_s | 0 \rangle \langle 0 | w_s \rangle X_s = I, \qquad (68)$$

$$\frac{1}{2} \sum_{s=1}^{m} \langle w_s | (|0\rangle\langle i| + |i\rangle\langle 0|) | w_s \rangle \mathrm{Tr}[X_s D_j] = \delta_i^j. \qquad (69)$$

Estimator attaining $S[P1, W_R] \geq S(P1)$

V: constructing W_R

W_R : finite subset of the hypersphere



δ is the covering radius of the hypersphere

$\delta = O(\sqrt{d}/|W_R|^{2/d})$

optimal solution $X_1^*, \ldots, X_m^*$

0th component of vector w_s

POVM $\{M_s\}$

$$M_s := |w_s^0|^2 X_s^*$$

estimator $(\{M_s\}, \hat{\theta})$

$$\hat{\theta}^i(s) := \frac{w_s^i}{w_s^0} + \theta^i$$

dual problem $[D1, \mathcal{W}_R]$ optimal value $S[D1, \mathcal{W}_R] := \sum_i (a^*)^i_i + \mathrm{Tr} S^*$

$$\begin{aligned} \underset{(a,S)\in\mathbb{R}^{d\times d}\times\mathcal{T}_{sa}(\mathcal{H})}{\text{maximize}} \quad & \sum_i a^i_i + \mathrm{Tr} S \\ \text{subject to} \quad & \sum_{s=1}^{m} |s\rangle\langle s| \otimes \langle w_s|\Pi(a,S)|w_s\rangle \geq 0 \end{aligned}$$

$$\Pi(a,S) := G\otimes\rho - \frac{1}{2}\Big(\sum_{1\leq i,j\leq d} a^j_i (|0\rangle\langle i| + |i\rangle\langle 0|)\otimes D_j\Big) - |0\rangle\langle 0|\otimes S. \qquad (57)$$

$$C_2(a) := \frac{1}{2}\left\|\sum_j \Big(\rho^{-1/2}\big(\sum_{j'} a^{j'}_j D_{j'}\big)\rho^{-1/2}\Big)^2\right\|^{1/2}$$

$$X^* := \Pi(a^*, S^*)$$

$$\kappa := -\min_{v\in\mathcal{H}:\|v\|=1}\ \min_{y:\|y\|\leq C_2(a^*)} \langle (1,y),v|X^*|(1,y),v\rangle$$

$$\delta(\mathcal{W}_R) := \max_{x\in\mathbb{R}^{d+1}:\|x\|=1}\ \min_{w\in\mathcal{W}_R} \||x\rangle\langle x| - |w\rangle\langle w|\|_1$$

Theorem 13.

$$\begin{aligned} S[D1,\mathcal{W}_R] &= S[P1,\mathcal{W}_R] \geq S(P1) = S(D1) \\ &\geq S[D1,\mathcal{W}_R] - n\|X^*\|(1+C_2(a^*)^2)\delta(\mathcal{W}_R). \end{aligned}$$

Idea: Given solution (a,S) to $[D1, \mathcal{W}_R]$ construct solution for D0. (non-trivial)

This gives a lower bound to S(D0)=S(P0)=S(P1)=C[G].

CONSTRUCTION OF $\mathcal{W}_R$

$$\mathcal{D}_{n,1} := \left\{\left(\cos\frac{2\pi j}{n}, \sin\frac{2\pi j}{n}\right)\right\}_{j=0}^{n-1}$$

$$|\mathcal{D}_{n,d}| = n^d \leq \frac{(2\pi)^d d^{d/2}}{\delta^d}$$

$$\mathcal{D}_{n,d} := \left\{\left(\cos\frac{2\pi j}{n} v, \sin\frac{2\pi j}{n}\right) \middle| v \in \mathcal{D}_{n,d-1,R}, j = 0, \ldots, n-1\right\}$$

We use this theoretically to bound δ.

A. Quantum t-design

$$\sum_{w\in\mathcal{W}_R} \frac{1}{|\mathcal{W}_R|} |w\rangle\langle w|^{\otimes t} = \int |x\rangle\langle x|^{\otimes t} \mu_{\mathcal{R}}(dx)$$

B. Spherical t-design

We use this numerically.

$d = 2$

$\mathcal{W}_R$ as $\cup_{j=0}^{k}\{(\cos\frac{\phi_0 j}{k}, \sin\frac{\phi_0 j}{k} y)\}_{y\in\mathcal{S}_j}$

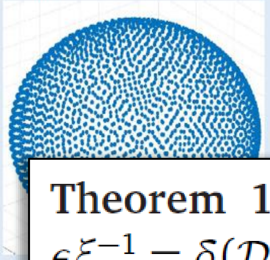
$\tan\phi_0 = C_2(a^*)$

choose $\mathcal{S}$ as

$\{(\cos\frac{2\pi l}{N}, \sin\frac{2\pi l}{N})\}_{l=1}^{N}$

V: constructing W_R

W_R : finite subset of the hypersphere



δ is the covering radius of the hypersphere

Theorem 16. *Suppose that we choose $\mathcal{W}_R$ as $\mathcal{D}_{n,d}$ with $\epsilon\xi^{-1} = \delta(\mathcal{D}_{n,d})$. Then, the calculation complexity of the tight CR bound for probe states of size n within additive error ϵ is*

$$O\Big(\frac{\xi^d d^{d/2}}{\epsilon^d}(n^6 + d^2 n^3 + d^4 n^2)\Big). \tag{114}$$

Algorithm `makeWR`

$\mathcal{W}_R = \texttt{makeWR}(N)$

1. Set $N = 70, k = 100$ and $\phi_0 = 1.2$.
2. Set $\mathcal{W}_R = \emptyset$
3. for idx $= 1 : (k+1)$
4. Set currN $= \max(\lceil (\text{idx}/k)^{1/4} N \rceil, 20)$
5. Set calS $= \texttt{circlepoints}(\text{currN})$
6. Set $j = \text{idx} - 1$
7. Set $\phi = \phi_0 j / k$
8. Set **1** as a column vector of ones that has the same number of rows as calS
9. Set W as the matrix with three columns $[\cos(\phi)\mathbf{1}, \sin(\phi)\text{calS}]$ with N rows.
10. Add every row of W into the set $\mathcal{W}_R$.

Algorithm `circlepoints`

calS $= \texttt{circlepoints}(N)$

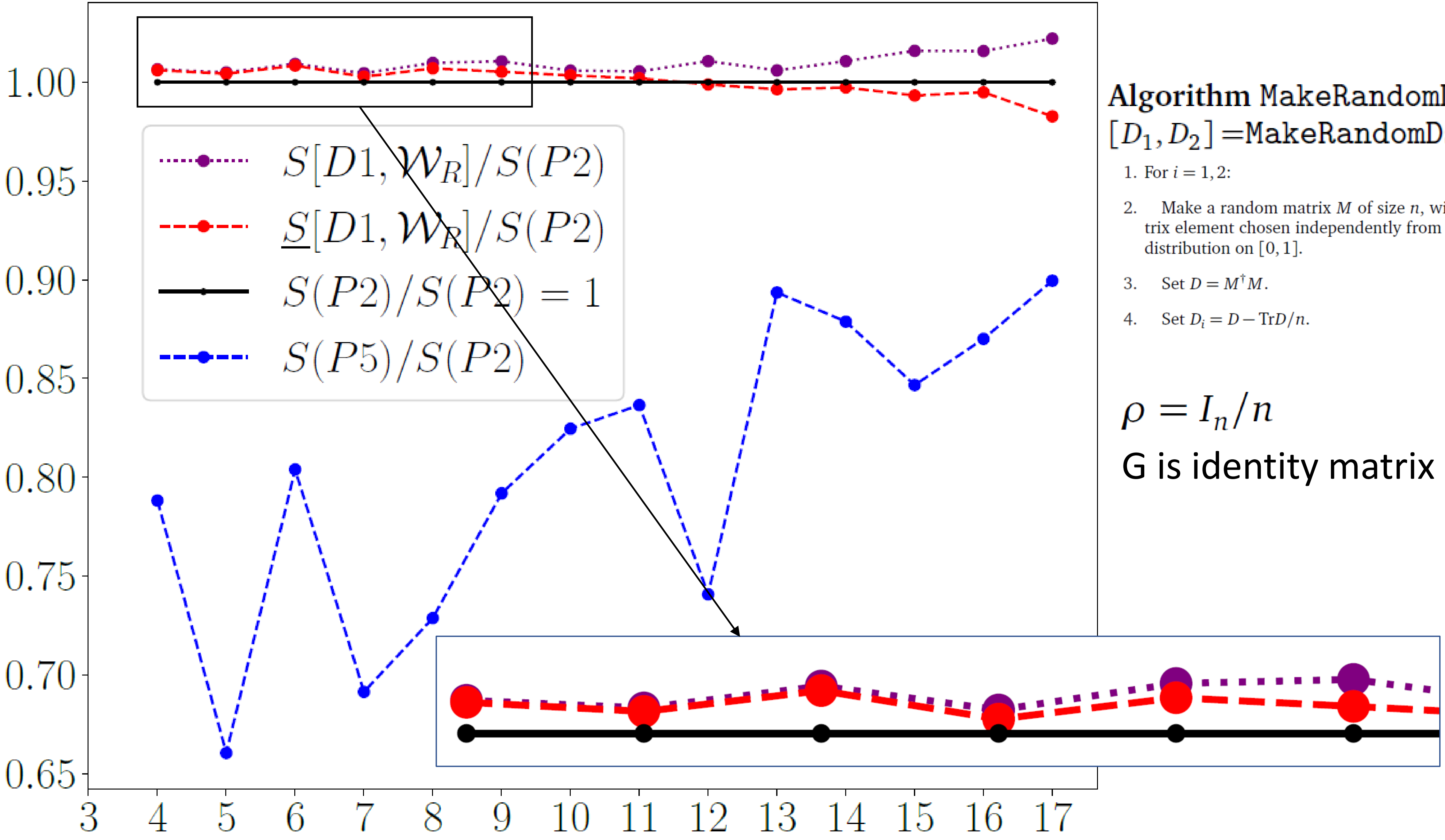
1. Set calS $=$ a matrix with N rows and two columns, with every entry equal to zero.
2. For $j = 1 : N$
3. Set $\theta = 2\pi j / N$
4. Set calS$(j, 1) = \cos(\theta)$
5. Set calS$(j, 2) = \sin(\theta)$

approximate κ

1. Set myrange $= \{-c_2 + 2c_2 j/999 | j = 0, \ldots, 999\}$. (We can implement this in MatLab using linspace$(-c_2, c_2, 1000)$).
2. Set $\kappa = -\infty$.
3. For x in myrange:
4. For y in myrange:
5. Set $w = (1, x, y)$ as a column vector.
6. If $\|(x, y)\| \leq c_2$:
7. Set $m = \lambda_{\min}(\langle w | X^* | w \rangle)$, where $\lambda_{\min}(\cdot)$ gives the minimum eigenvalue of a Hermitian matrix.
8. If $m < -\kappa$, set $\kappa = -m$.

$$\underline{S}[D1, \mathcal{W}_R] = \text{Tr}a^* + \text{Tr}S^* - n\kappa$$

(a*,S*) is optimal solution of [D1,W_R]



Algorithm `MakeRandomDs`

$[D_1, D_2]$ = `MakeRandomDs`(n)

1. For $i = 1, 2$:
2. Make a random matrix M of size n, with each matrix element chosen independently from the uniform distribution on $[0, 1]$.
3. Set $D = M^\dagger M$.
4. Set $D_i = D - \mathrm{Tr} D / n$.

$\rho = I_n / n$

G is identity matrix

APPLICATIONS

A. Learning parameters of Hamiltonian models

$H = \sum_k a_k P_k$ estimate the parameters $a_1, \ldots, a_d$

$$\mathcal{L}_{a,b}(\rho) = -i[H_{a,b}, \rho] + \sum_i \gamma_j \left(L_j \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, \rho\} \right)$$

$$\mathcal{M} = \{\exp(\mathcal{L}_{a,b}(\rho)) : a, b \in \mathbb{R}\}$$

B. 3D field sensing

$$H = \sum_{j=1}^{n} (xX_j + yY_j + zZ_j)$$

Same form as in A

$$\mathcal{M} = \{e^{\mathcal{L}}(\rho) : x, y, z \in \mathbb{R}\}$$

II Tight CR bound C[G]

C[G] = S(D0) [Ref 2]

Feasibility of (a,S) in **D0**

Needs Eq. (23) to hold for all x

Duality

III: Conic programming

Thm 2: **C[G] = S(P1)**

Lem 3: **C[G]** = C[G,d(d+1)n²/2]

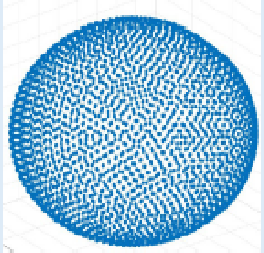
Thm 4: **NH** bound = S(P2),
SLD bound = S(P4)

Thm 5: **HCR** bound = S(P5)

Thm 6: Strong duality: S(Pi) = S(Di)

V: constructing W_R

W_R : finite subset of the hypersphere



δ is the covering radius of the hypersphere

$\delta = O(\sqrt{d}/|W_R|^{2/d})$

IV: Calculating S(P1)

IVA: **upper bound** via SDP

SDP depends on set W_R

Primal SDP: **[P1, W_R]**

Dual SDP: **[D1, W_R]**

IVB: **measurement strategy**

Thm 9: Estimator attaining **S(P1)**

IVC: **lower bound**

Thm 11:

S[P1, W_R] = S[D1, W_R]

S[D1, W_R] ≥ **S[P1] ≥ S[P1, W_R] - n κ**

κ depends on the optimal solution $X^* = \Pi(a^*, S^*)$ of **[D1, W_R]**.

IVD: Evaluation of **κ** (can approach 0)

Lem 12:

κ ≤ function of norm of **X*** and covering radius δ

Thm 13: Combine Lem 12 and Thm 11

Lem 14: Another upper bound on **κ**

Appendices

A: Application of **C[G] = S(D0)**

AA: 1-parameter case, exact solution.

AB: qubit case, exact solution.

B: Linear Programming with general cone

C: Proof of Thm 6

D: Proof of Thm 7

E: Evaluation of **κ** used in Sec V

F: Proof of Thm 17

VI: complexity of approximating C[G] within additive error ϵ

Lem 15: $C_2(a) \leq ||a||\, C_1$

Thm 16: Complexity = O(poly(n/ϵ))

Thm 17: $||\Pi(a^{**}, S^{**})||, ||a^{**}|| \leq$ function of *G* and SLD matrix

VII: numerical lower bound for S(P1)

Algo `makeWR`: makes **W_R**

Use Thm 11, approximate κ numerically

Compare numerical upper, lower bounds to S(P1) to S(P2) and S(P4) (see Fig 5,6)

Wrapping up

VIII: applications

A learning Hamiltonian parameters

B 3D field sensing

IX: discussions