Exact and Local Compression of Quantum Bipartite States

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Quantum data compression

Data compression is one of most fundamental information processing.

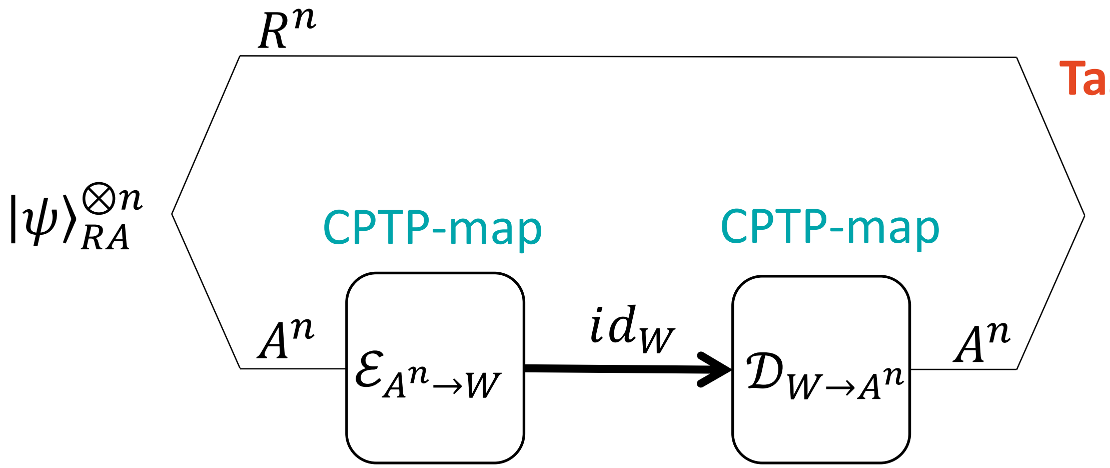
Noiseless coding

- **Quantum pure information source:** $\{p_x, |\phi_x\rangle\}_{x=1}^{|\chi|}$

Message: $\chi = \{1,2,\dots,|\chi|\}$

Averaged state: $\rho_A = \sum_{x=1}^{|\chi|} p_x |\phi_x\rangle\langle\phi_x|_A$

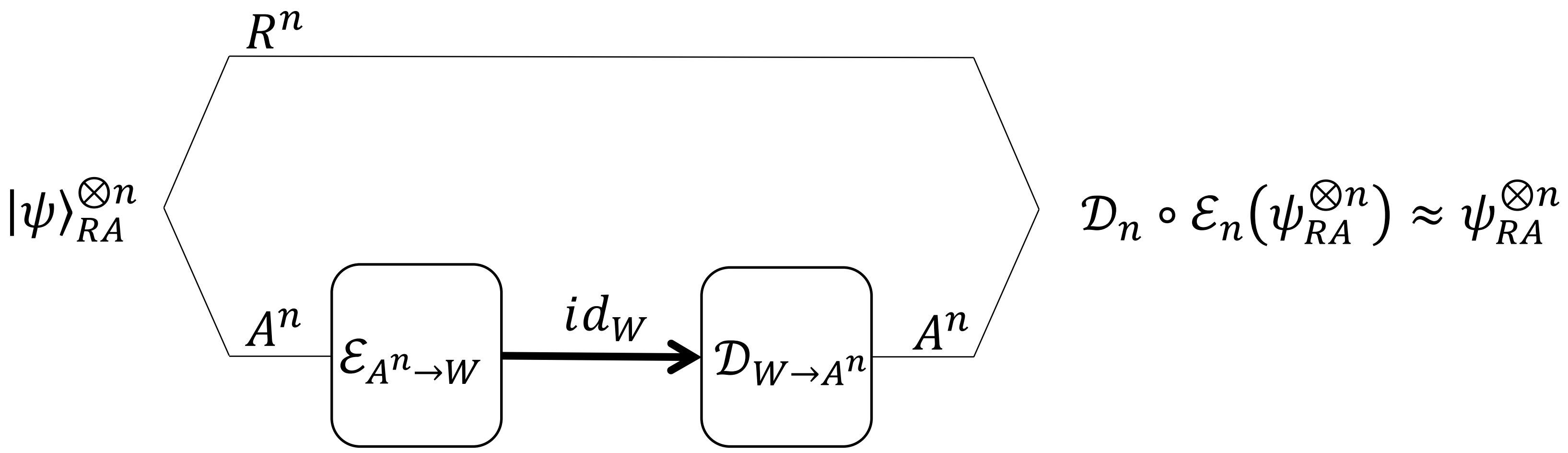
Purified source: $|\psi\rangle_{RA} := \sum_{x=1}^{|\chi|} \sqrt{p_x}|x\rangle_R|\phi_x\rangle_A$



Task: minimize $\dim\mathcal{H}_W$ under

$$\mathcal{D}_n \circ \mathcal{E}_n(\psi_{RA}^{\otimes n}) \approx \psi_{RA}^{\otimes n}$$

Noiseless coding theorem



Quantum noiseless channel coding theorem [Schumacher, '95]:

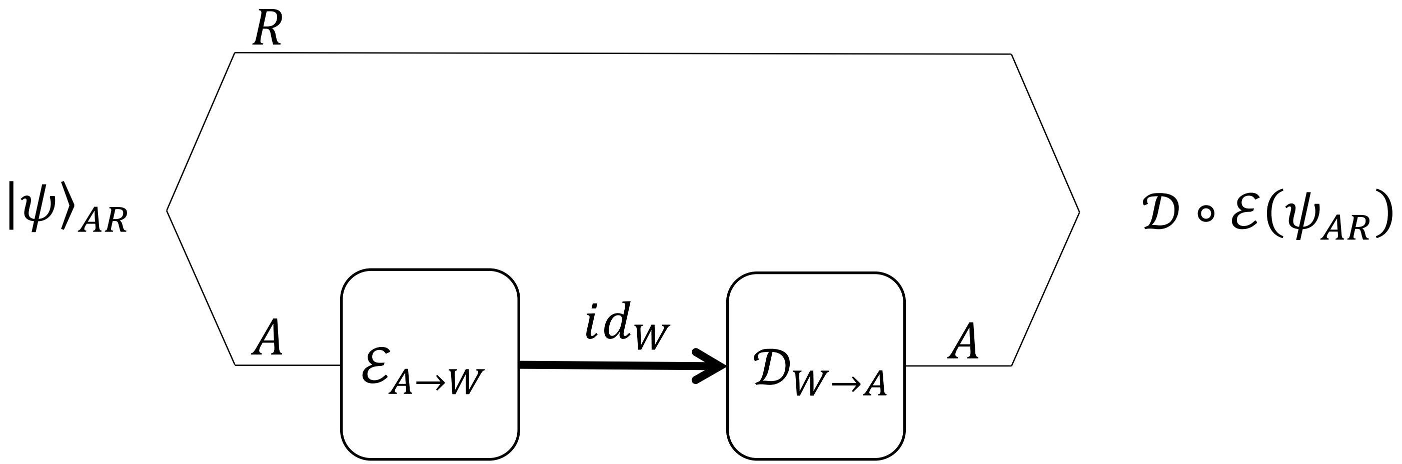
Asymptotically optimal dimension of W:

$$\lim_{n\to\infty} \frac{1}{n} \log(\dim \mathcal{H}_W) = S(\rho_A).$$

$S(\rho_A) \coloneqq -\mathrm{tr}\rho_A \log_2 \rho_A$ von Neumann entropy

One-shot data compression

- What if we can only use **one** state?

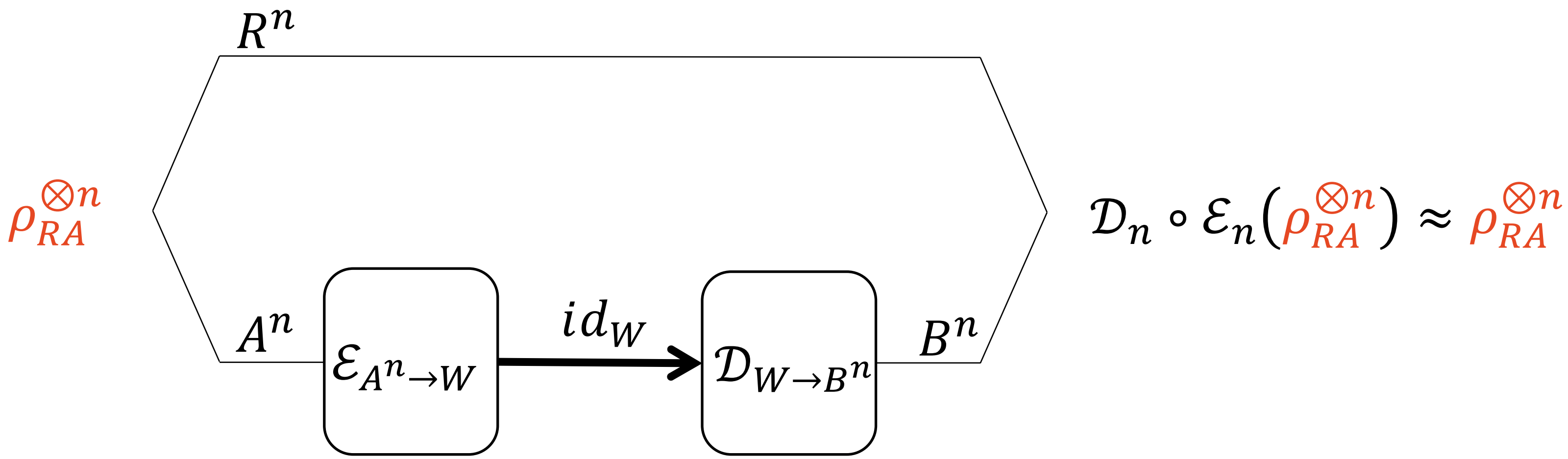


Exact $\mathcal{D}\circ\mathcal{E}(\psi_{AR}) = \psi_{AR}$ ➡ $\dim\mathcal{H}_W = \mathrm{rank}\rho_A$ **No nontrivial compression**

Approximate $\mathcal{D}\circ\mathcal{E}(\psi_{AR}) \approx \psi_{AR}$ ➡ **[Berta,’08][Datta, Hsieh, Wilde, ‘12],…**

Mixed state source (asymptotic)

- What if we consider **mixed state** source?



The optimal dimension of W is given by **Koashi-Imoto decomposition** **[Koashi, Imoto, '02]**

$$\rho_{RA} = \bigoplus_i p_i \, \rho_{RA_i^L} \otimes \omega_{A_i^R}$$

$$\lim_{n\to\infty} \frac{1}{n} \log(\dim \mathcal{H}_W) = S\left(\sum_i p_i \rho_{RA_i^L}\right) \leq S(\rho_A)$$

- R is classical **[Koashi, Imoto, '01]**
- R is quantum **[Khanian,Winter, '20]**

Koashi-Imoto (KI) decomposition

[Koashi, Imoto, '02][Hayden, et al., '04]

Consider a bipartite state $\rho_{RA} \in \mathcal{B}(\mathcal{H}_R \otimes \mathcal{H}_A)$, s.t., $\rho_A > 0$.

There exists a decomposition $\mathcal{H}_A \cong \bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R}$ s.t.,

$$\rho_{RA} = \bigoplus_i p_i \underbrace{\rho_{RA_i^L}}_{\text{Quantumly correlated}} \otimes \underbrace{\omega_{A_i^R}}_{\text{Classically correlated}}$$

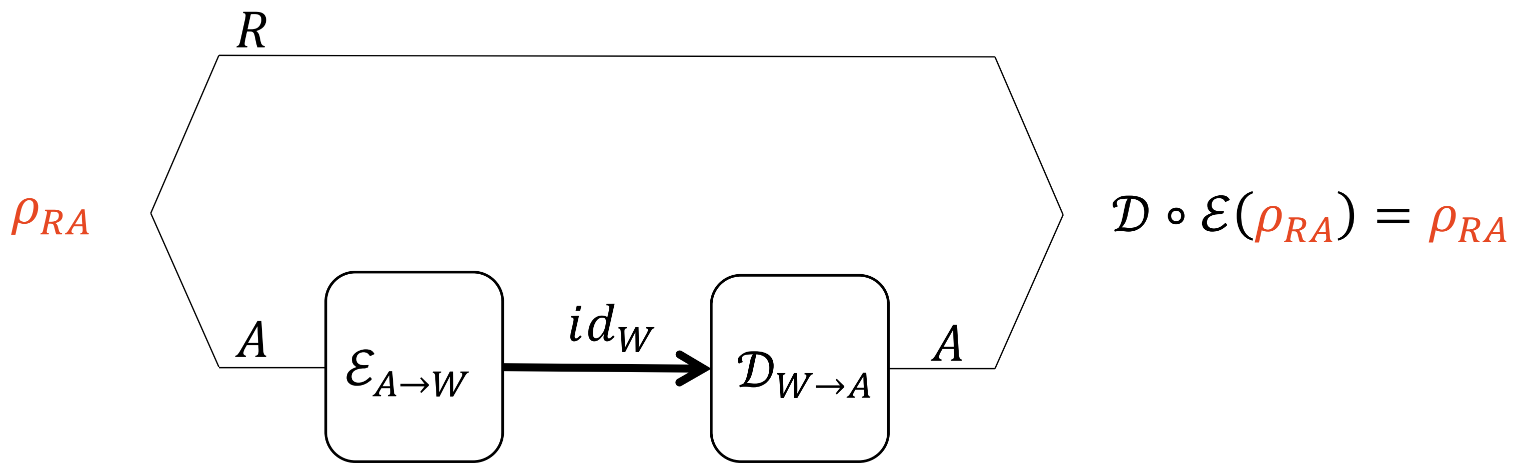
and for any CPTP-map $\Lambda_A(\cdot) = \mathrm{tr}_E\left(V_{A\to AE} \cdot (V_{A\to AE})^\dagger\right)$ satisfying $\Lambda_A(\rho_{RA}) = \rho_{RA}$,

$V_{A\to AE}$ is decomposed as

$$\begin{cases} V_{A\to AE} = \bigoplus_i I_{A_i^L} \otimes V_{A_i^R \to A_i^R E}, \\ \mathrm{tr}_E\left(V_{A_i^R\to A_i^R E}\,\omega_{A_i^R}\left(V_{A_i^R\to A_i^R E}\right)^\dagger\right) = \omega_{A_i^R}, \ \forall i. \end{cases}$$

Main problem in this talk

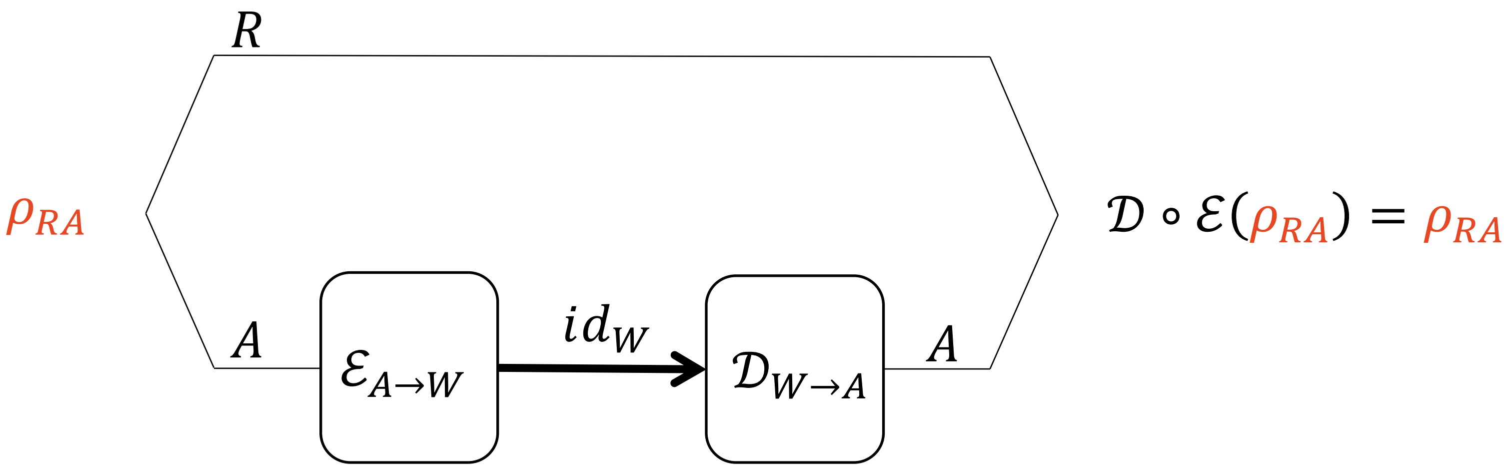
- **One-shot, exact** compression of **mixed** quantum source



Question

What is the minimum dimension of $\mathcal{H}_W$?

Minimum dimension



The minimum achievable compression is also given by KI-decomposition.

$$\mathcal{H}_A \cong \bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R} \quad \Rightarrow \quad \mathcal{H}_W = \bigoplus_i \mathcal{H}_{A_i^L}$$

However, how to calculate $\dim\mathcal{H}_W$?

The known algorithms to calculate KI-decomposition **[Koashi, Imoto, ‘02] [Yamasaki, Murao, ‘19]** are hard to implement.

Main results

Consider any bipartite state $\rho_{RA} \in \mathcal{B}(\mathcal{H}_R \otimes \mathcal{H}_A)$.

We restrict $\mathcal{H}_A \coloneqq \mathrm{supp}(\rho_A)$ w.l.o.g. $(\rho_A > 0)$.

- **Results (details in later)**

We developed a way to calculate a pure state $|\tilde{C}\rangle_{A\bar{A}A_1\overline{A_1}}$ s. t.,

$$\dim\mathcal{H}_W = \sum_i \dim\mathcal{H}_{A_i^L} = \mathrm{rank}(\tilde{C}_{\bar{A}}) \ .$$

$(A \cong \bar{A} \cong A_1 \cong \overline{A_1})$

$\tilde{C}_{\bar{A}} \coloneqq \mathrm{tr}_{AA_1\overline{A_1}}|\tilde{C}\rangle\langle\tilde{C}|$

We also found bounds which are easier to calculate

$$\sqrt{\mathrm{rank}(C_{AA_1})} = \sqrt{\sum_i \left(\dim\mathcal{H}_{A_i^L}\right)^2} \le \dim\mathcal{H}_W \le \mathrm{rank}(C_{AA_1}).$$

Exact diagonalization of $d_A^2 \times d_A^2$ matrices

Classical case

$$\rho_{RA} := \sum_{r,a} p(r,a)|r,a\rangle\langle r,a|$$

$$\mathcal{H}_A \cong \bigoplus_i \mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R}, \quad \dim\mathcal{H}_{A_i^L} = 1, \forall i.$$

A nontrivial compression exists iff $\exists a, a'$ s.t.

$$p(r|a) = p(r|a'), \forall r.$$

$p(r,a) = p(r)p(a)$ independent

$p(r|a_1, a_2) = p(r|a_1)$ Markov

KI-decomposition is known as **Fisher–Neyman factorization theorem** [Fisher, '22][Neyman, '36]

Method

Minimal sufficient subalgebra

The condition $\mathcal{D}_{W\to A}\circ\mathcal{E}_{A\to W}(\rho_{RA})=\rho_{RA}$ is equivalent to the following **[Hayden, et al., '04]:**

$$\mathcal{D}_{W\to A}\circ\mathcal{E}_{A\to W}(\mu_A)=\mu_A, \forall\mu\in\mathcal{S}, \qquad \mathcal{S}:=\left\{\mu_A=\frac{\mathrm{tr}_R(M_R\rho_{RA})}{\mathrm{tr}(M_R\rho_R)}\middle|0\le M_R\le I_R\right\}.$$

[Jenčová&Petz, '06]

▶ The KI-decomposition of ρ_{RA} is originated in the **minimum sufficient subalgebra** of $\mathcal{S}$.

[Jenčová, '12]

$$\mathcal{M}_A^S:=\mathrm{Alg}\{\mu_A^{it}\rho_A^{-it},\mu\in\mathcal{S},t\in\mathbb{R}\}=\mathrm{Alg}\left\{\rho_A^{it-1/2}\mu_A\rho_A^{-it-1/2},\mu\in\mathcal{S},t\in\mathbb{R}\right\}$$

want to know the dimension

$$\mathcal{H}_A\cong\bigoplus_i\mathcal{H}_{A_i^L}\otimes\mathcal{H}_{A_i^R} \iff \mathcal{M}_A^S\cong\bigoplus_i\mathrm{Mat}\left(\mathcal{H}_{A_i^L},\mathbb{C}\right)\otimes I_{A_i^R}$$

Strategy to calculate the dimension

$$\mathcal{M}_A^S \cong \bigoplus_i \mathrm{Mat}\left(\mathcal{H}_{A_i^L}, \mathbb{C}\right) \otimes I_{A_i^R}$$

$\left(\mathcal{M}_A^S\right)' \coloneqq \{X_A | [X_A, Y_A] = 0, Y_A \in \mathcal{M}_A^S\}$ the commutant of $\mathcal{M}_A^S$ $\cong \bigoplus_i I_{A_i^L} \otimes \mathrm{Mat}\left(\mathcal{H}_{A_i^R}, \mathbb{C}\right)$

- The conditional expectation $\mathbb{E}_A$ on $\left(\mathcal{M}_A^S\right)'$:

$$\mathbb{E}_A(\cdot) = \bigoplus_i \tau_{A_i^L} \otimes \mathrm{tr}_{A_i^L}(\Pi_i \cdot \Pi_i),$$

Π_i: the projection onto $\mathcal{H}_{A_i^L} \otimes \mathcal{H}_{A_i^R}$.

τ: completely mixed state

- Strategy

1. Obtain the Choi state of the conditional expectation $\mathbb{E}_A$ on $\left(\mathcal{M}_A^S\right)'$.
2. Extract information of $d_{A_i^L}$ from the Choi state.

Characterizing the subalgebra

- **Petz recovery map:** For a CPTP-map $\mathcal{T}$ and a state $\sigma > 0$, define

$$\mathcal{R}^{\sigma,\mathcal{E}}(\cdot) \coloneqq \sigma^{\frac{1}{2}}\mathcal{T}^{\dagger}\left(\mathcal{T}(\sigma)^{-1/2} \cdot \mathcal{T}(\sigma)^{-1/2}\right)\sigma^{\frac{1}{2}}.$$

- **Several definitions**

$J_{RA} \coloneqq \rho_A^{-1/2}\rho_{RA}\rho_A^{-1/2}$. $\Omega_{R\to A}^{\dagger}(X_R) \coloneqq \mathrm{tr}_R\left(J_{RA}(X_R^T \otimes I_A)\right)$. $\Omega_{A\to E} \coloneqq (\Omega_{A\to R})^c$. complementary channel

$\mathcal{T}_{A\to A} \coloneqq \mathcal{R}_{E\to A}^{\tau,\Omega} \circ \Omega_{A\to E}$. $Fix(\mathcal{T}_{A\to A}) \coloneqq \{X_A | \mathcal{T}_{A\to A}(X_A) = X_A\}$. fixed point algebra

Lemma $\left(\mathcal{M}_A^S\right)'$ is the largest subalgebra of $Fix(\mathcal{T}_{A\to A})$ that is invariant under $\Delta_\rho^t(\cdot) \coloneqq \rho_A^{it} \cdot \rho_A^{-it}$ for all $t \in \mathbb{R}$.

$$\rho_A^{-1/2}\mu_A\rho_A^{-1/2} = \frac{\mathrm{tr}_R(M_R J_{RA})}{\mathrm{tr}(M_R\rho_R)} \propto \Omega_{R\to A}^{\dagger}(M_R^T) \qquad X_A \in \left(\mathcal{M}_A^S\right)' \Leftrightarrow \left[X_A, \Delta_\rho^t \circ \Omega_{R\to A}^{\dagger}(M_R^T)\right] = 0, \forall M_R \geq 0.$$

CPTP-maps as a matrix (superoperator)

Lemma $(\mathcal{M}_A^S)'$ is the largest subalgebra of $Fix(\mathcal{T}_{A\to A})$ that is invariant under $\Delta_\rho^t(\cdot) \coloneqq \rho_A^{it} \cdot \rho_A^{-it}$ for all $t \in \mathbb{R}$.

linear map from A to A

operator on $A \otimes A_1$ $(A_1 \cong A)$

Superoperator of $\mathcal{T}_{A\to A}$: $E_{\mathcal{T}} = \bigoplus_\lambda \lambda P_\lambda$

$RL_A \coloneqq I_A \otimes \log\rho_{A_1}^T - \log\rho_A \otimes I_{A_1} = \bigoplus_\eta \eta Q_\eta$

generator of Δ_ρ^t

$(\mathcal{M}_A^S)'$ subalgebra

⇕

$V \coloneqq \bigoplus_\eta \mathrm{supp}(Q_\eta) \cap \mathrm{supp}(P_1)$

operator subspace

The superoperator of $\mathbb{E}_A$= the projector on V

$$P_V = 2\bigoplus_\eta Q_\eta (Q_\eta + P_1)^{-1} P_1$$

[Anderson&Duffin, '69]

⇔

The Choi state of $\mathbb{E}_A$

$$d_A \cdot C_{AA_1} \coloneqq (id_A \otimes \mathbb{E}_{A_1}) \textstyle\sum_{i,j} |ii\rangle\langle jj|_{AA_1}$$

$$= \textstyle\sum_{i,j} (I_A \otimes |i\rangle\langle j|_{A_1}) P_V (|i\rangle\langle j|_A \otimes I_{A_1})$$

From the Choi state to the dimension

- The Choi state of $\mathbb{E}_A$

$$C_{AA_1} = \bigoplus_i p_i \tau_{A_i^L} \otimes |\Psi_i\rangle\langle\Psi_i|_{A_i^R A_{1i}^R} \otimes \tau_{A_{1i}^L}$$

τ: completely mixed state

Ψ_i: maximally entangled state

$$p_i = \frac{d_{A_i^L} d_{A_i^R}}{d_A}$$

$$\text{rank}(C_{AA_1}) = \sum_i \dim\left(\mathcal{H}_{A_i^L}\right) \dim\left(\mathcal{H}_{A_{1i}^L}\right) = \sum_i \left(\dim\left(\mathcal{H}_{A_i^L}\right)\right)^2$$

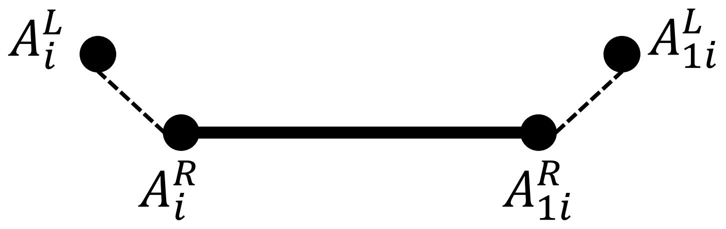
Meanwhile, $\dim\mathcal{H}_W = \sum_i \dim\left(\mathcal{H}_{A_i^L}\right).$

$$\sqrt{\text{rank}(C_{AA_1})} = \sqrt{\sum_i \left(\dim\mathcal{H}_{A_i^L}\right)^2} \leq \dim\mathcal{H}_W \leq \text{rank}(C_{AA_1}).$$

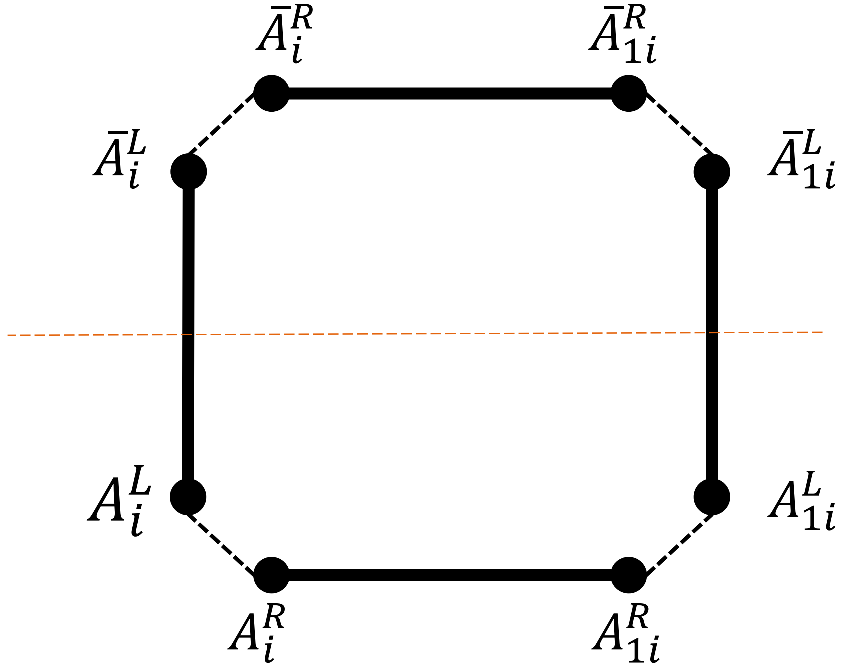
From the Choi state to the dimension

$|C\rangle_{A\bar{A}A_1\overline{A_1}}$: the canonical purification of the Choi state C_{AA_1}

$$C_{AA_1} = \bigoplus_i p_i \tau_{A_i^L} \otimes |\Psi_i\rangle\langle\Psi_i|_{A_i^R A_{1i}^R} \otimes \tau_{A_{1i}^L} \longrightarrow |C\rangle_{A\bar{A}A_1\bar{A}_1} = \sum_i \sqrt{p_i} |\Psi_i\rangle_{A_i^L \bar{A}_i^L} |\Psi_i\rangle_{A_i^R A_{1i}^R} |\Psi_i\rangle_{\bar{A}_i^R \bar{A}_{1i}^R} |\Psi_i\rangle_{A_{1i}^L \bar{A}_{1i}^L} .$$



C_{AA_1}

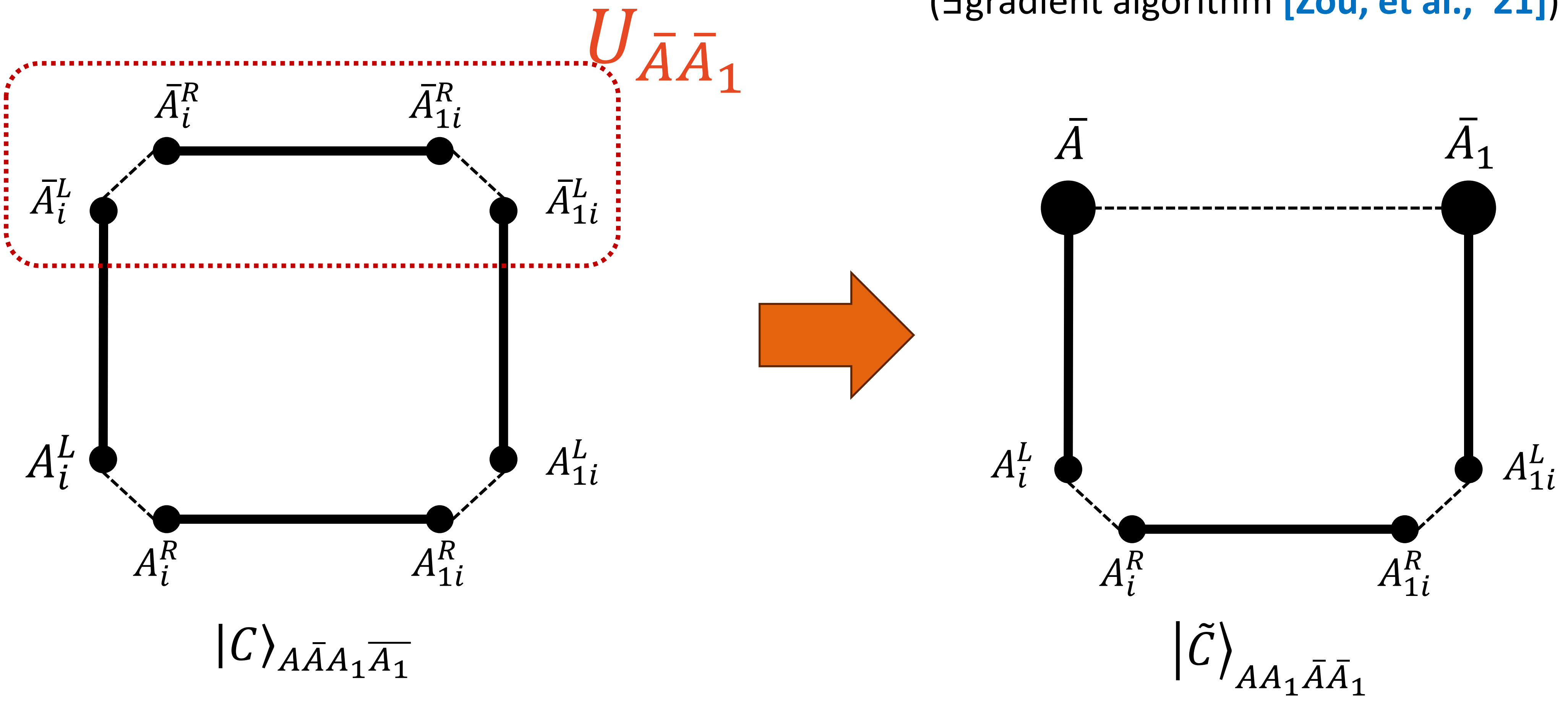


$|C\rangle_{A\bar{A}A_1\overline{A_1}}$

From the Choi state to the dimension

Apply a unitary on $\bar{A}\bar{A}_1$ to minimize the entanglement between $A\bar{A}$: $A_1\bar{A}_1$.

(∃gradient algorithm [Zou, et al., '21])



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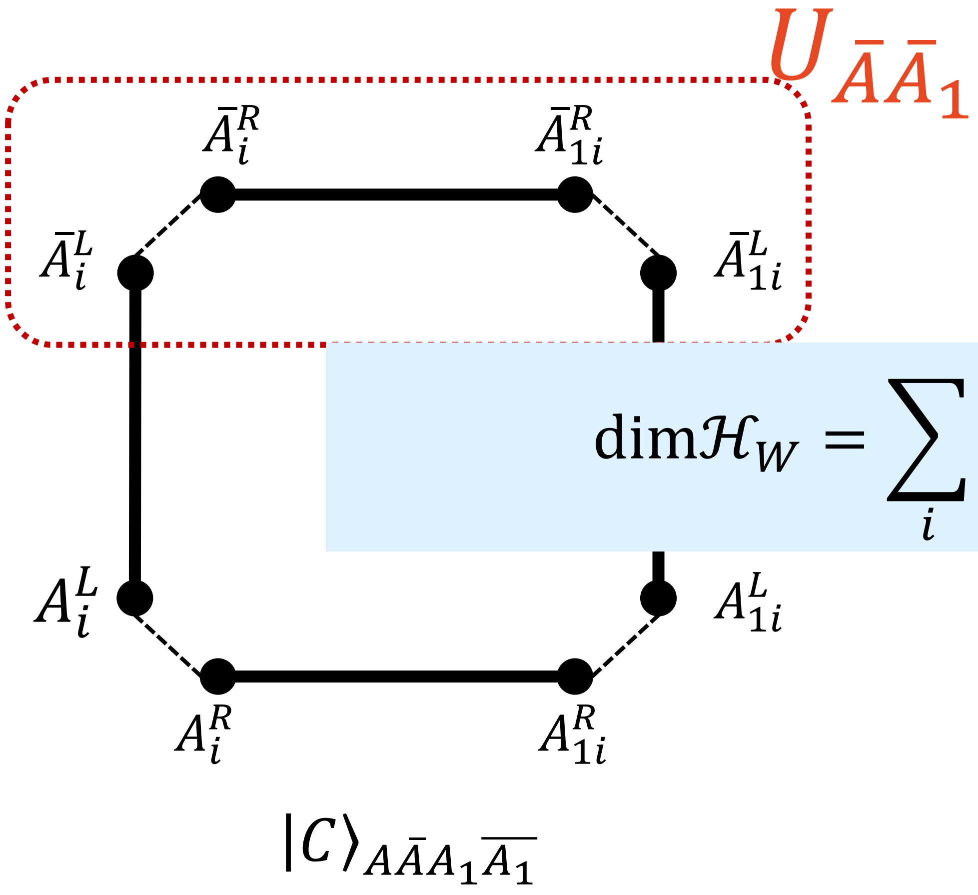
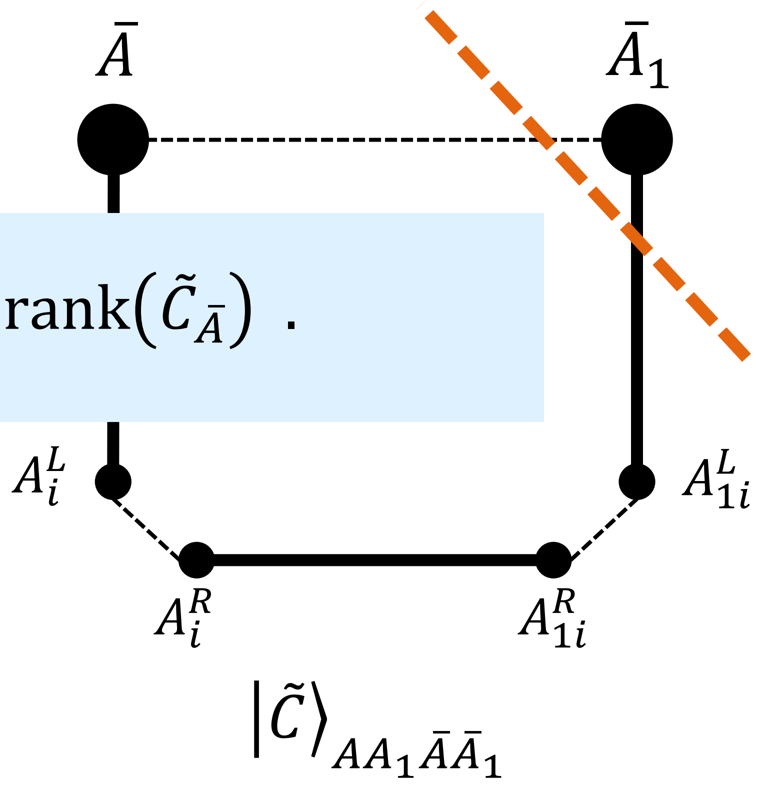


$$\dim\mathcal{H}_W = \sum_i \dim\mathcal{H}_{A_i^L} = \mathrm{rank}\left(\tilde{C}_{\bar{A}}\right) \ .$$



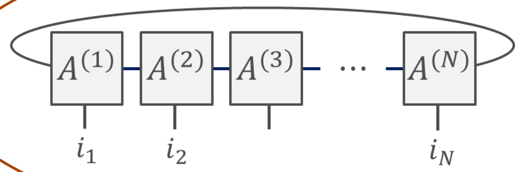
Future direction (Motivation)

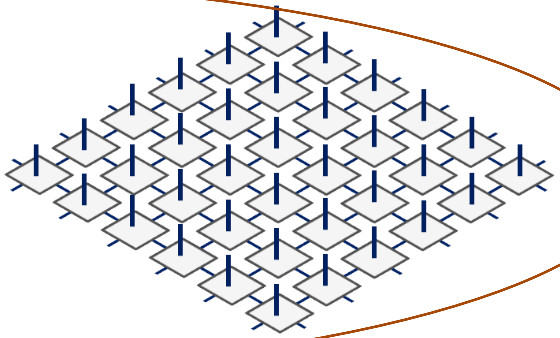
Tensor Networks

$$|\psi\rangle = \sum_{i_1,..,i_N} c_{i_1 i_2 \ldots i_N} |i_1 \ldots i_N\rangle$$

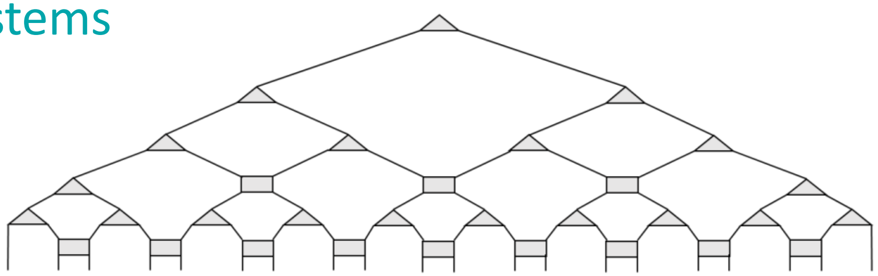
Gapped systems

$c_{i_1 i_2 \ldots i_N} =$

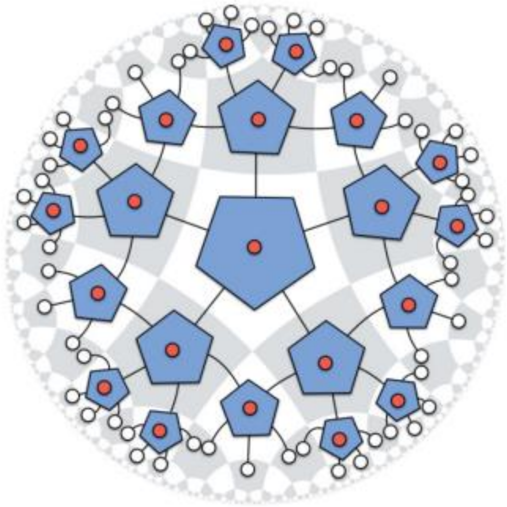




Critical systems



High-energy physics

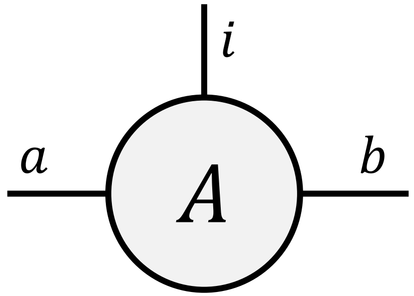


Various applications in quantum physics

[from arXiv:1802.01040]

Contraction rule

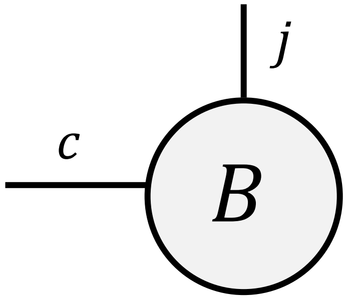
- Open leg = index of the tensor



$$A^i_{ab} \in \mathbb{C}$$

$$i = 1, \dots, d$$
$$a, b = 1, \dots, D$$

$$A^i := \left(A^i_{ab}\right)\text{: matrix}$$

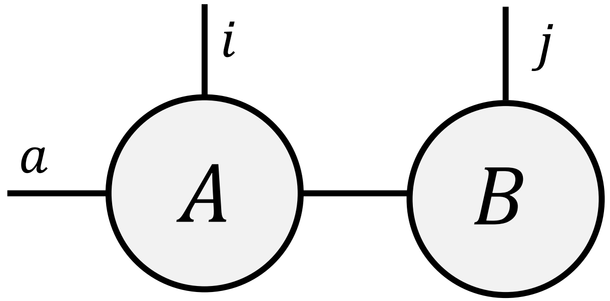


$$B^j_c \in \mathbb{C}$$

$$i = 1, \dots, d'$$
$$c = 1, \dots, D$$

$$|B^j\rangle := \left(B^j_c\right)\text{: vector}$$

- Connected leg = sum over the index



$$\sum_{b=1}^{D} A^i_{ab} B^j_b \in \mathbb{C},$$

vector

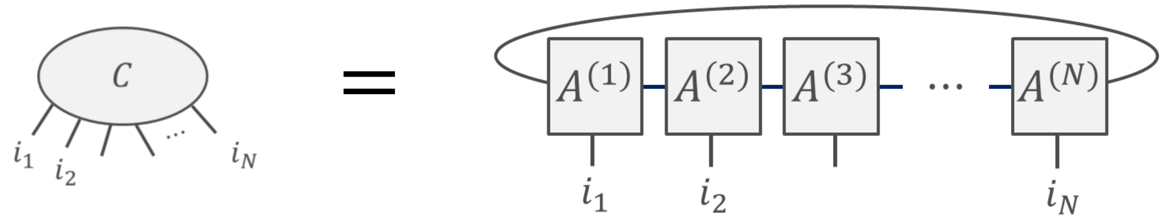
$$A^i|B^j\rangle = \left(\sum_{b=1}^{D} A^i_{ab} B^j_b\right)$$

Matrix Product states (MPS)

$$|\psi\rangle = \sum_{i_1,\ldots,i_N} c_{i_1\ldots i_N} |i_1 i_2 \ldots i_N\rangle \in \mathbb{C}^{d^{\otimes N}}$$

MPS $$|\psi\rangle = \sum_{i_1,..,i_N} \mathrm{Tr}\left(A^{(1)}_{i_1} A^{(2)}_{i_2} \ldots A^{(3)}_{i_N}\right) |i_1 \ldots i_N\rangle$$

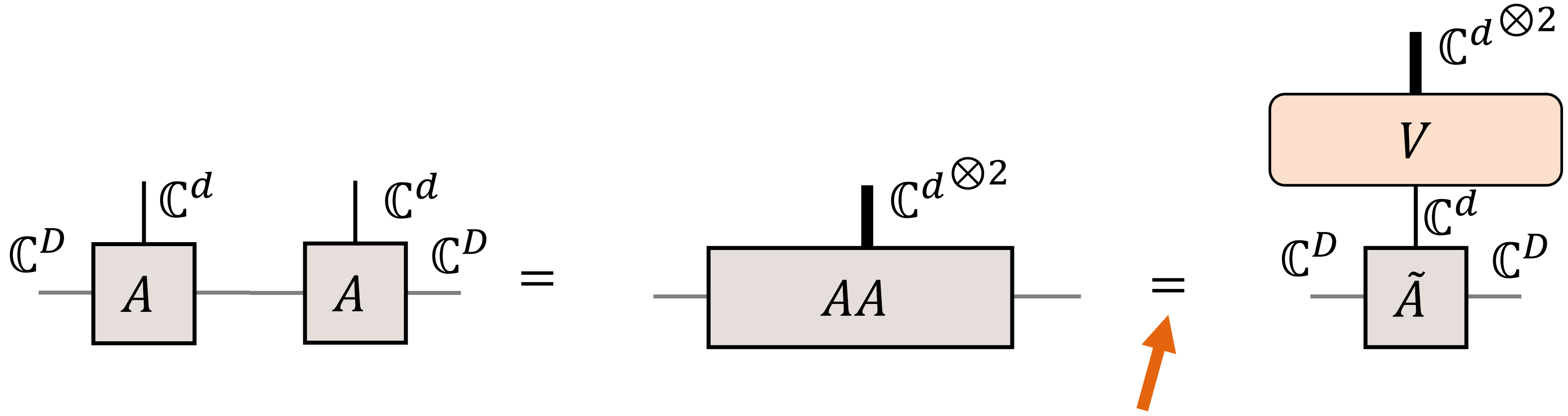
$A^{(j)}_{i_k}$: $D \times D$ matrix (for each i_k, j)



- **The number of parameters needed to specify a MPS** $= \boldsymbol{dND^2 \ll d^N}$
- **Always satisfies an area law of entanglement:** $S(X)_\psi := -\mathrm{Tr}\rho_X \log\rho_X \leq \log D$

Renormalization Group flow of MPS

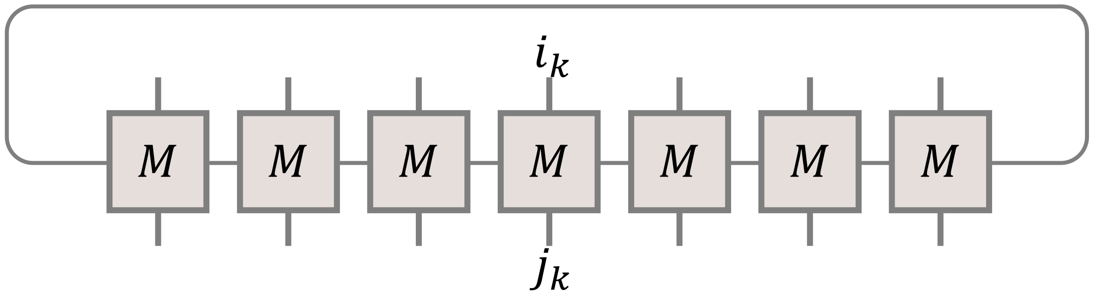
- **MPS has a physically applicable coarse-graining operation.**



Use the polar decomposition of tensor $AA = V\tilde{A}$ $(d > D^2\ w.l.o.g.)$.

- **The RG-fixed point is achieved by iteration**
- **The RG-fixed point is useful to characterize e.g., quantum phases [Schuch et al., '11]**

Matrix Product Density Operators (MPDO)



$$\rho_{MPDO} = \sum_{\boldsymbol{i},\boldsymbol{j}} \mathrm{Tr}\left(M_{i_1 j_1} M_{i_2 j_2} \ldots M_{i_N j_N}\right) |i_1 i_2 \ldots i_N\rangle\langle j_1 j_2 \ldots j_N|$$

$M_{i_k j_k}: D \times D$ matrix (for each i_k, j_k)

- A natural generalization of Matrix Product States to 1D **mixed states**.
- A good ansatz for thermal states and steady states in 1D systems.

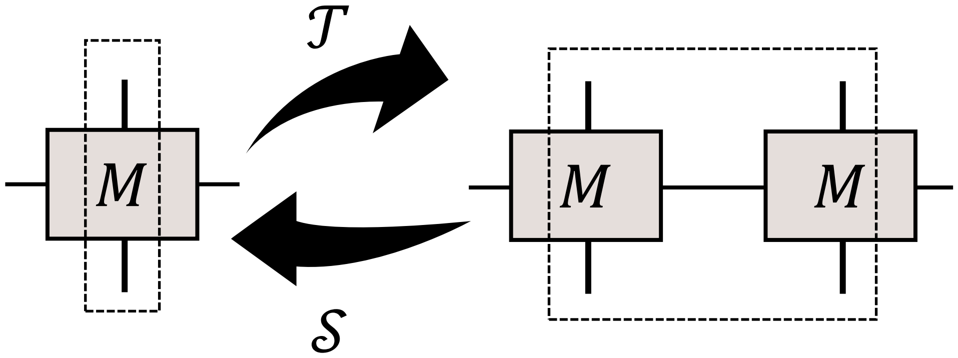
➢ **Any Gibbs states of 1D local Hamiltonian can be approximated by a MPDO [Hastings '06].**

$$\text{MPDO} \supseteq \rho_{Gibbs} = \frac{1}{Z} e^{-\beta \sum_i h_{i,i+1}}$$

Renormalization fixed-points of MPDO

[Cirac, et al., '17] Introduces "renormalization fixed-point" MPDOs.

A MPDO is a fixed-point MPDO if there are CPTP-maps $\mathcal{S}, \mathcal{T}$ such that



Theorem [Cirac, et al., '17] :
If ρ is a **fixed-point MPDO** and is simple*, then ρ has a **nearest-neighbor commuting parent Hamiltonian.**

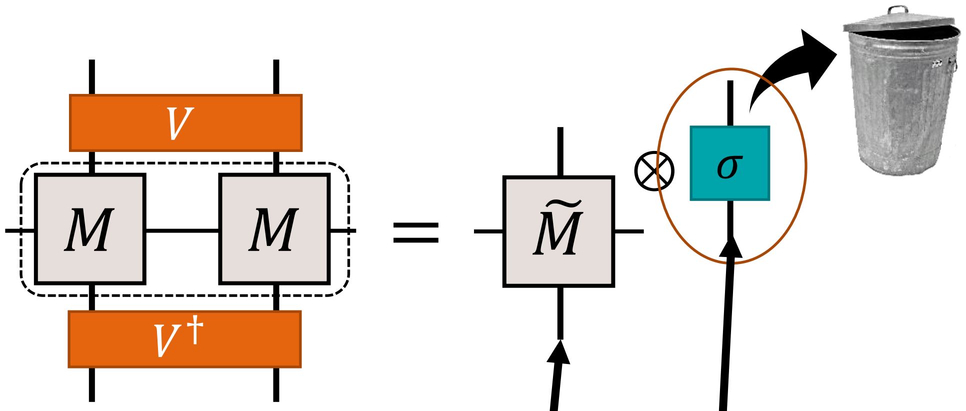
*M is simple if none of the canonical blocks is traceless.

Caveat: A notion of **renormalization flow is missing** for these "fixed-points".

Exact (reversible) RG-flow of MPDO?

- We need to establish RG-flow (coarse-graining) for MPDOs

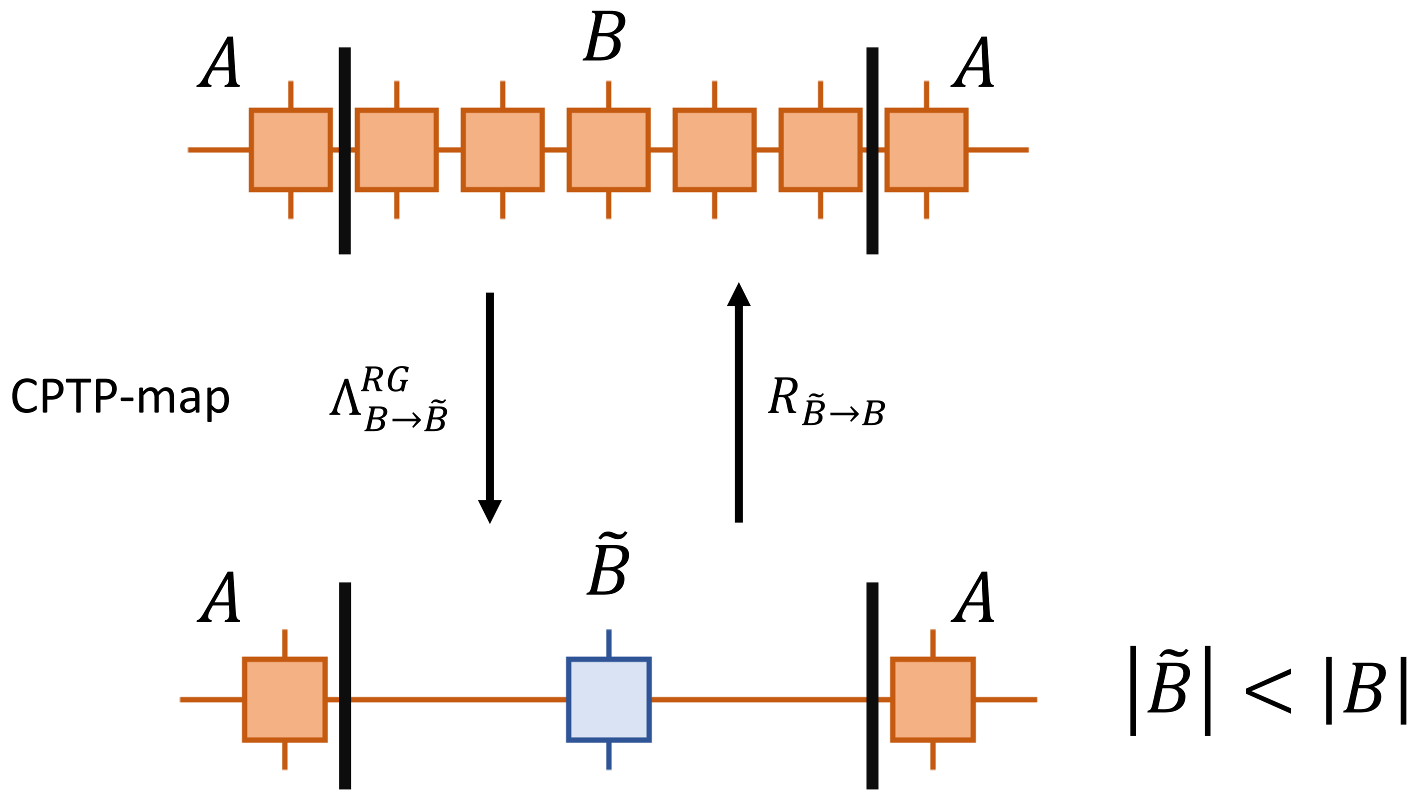
Unlike RG-flow for MPS (which is well-defined), one needs to reduce the entropy to keep the local dimension constant.



This is exactly given by **the Koashi-Imoto decomposition!**

$$\rho_{RA} = \bigoplus_i p_i \, \rho_{RA_i^L} \otimes \omega_{A_i^R}$$

Exact (reversible) RG-flow of MPDO?



If $|\tilde{B}|$ can chosen to be O(1), then we obtain the desired RG-flow.

Numerical test is on-going

Summary & Discussion

▶ Summary

- We have studied **one-shot** and **exact** data compression of **mixed** quantum source
- We have obtained **a formula for the minimum achievable dimension**

▶ Future direction

Many-boy physics

- Application to **tensor-network states?**

Quantum information

- How about one-shot **approximate** scenario?
- More sophisticated algorithm? Relation to entropic quantities?