Quantum Master Equation and Geometric Quantization

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Outline

In this talk, I will introduce some of the joint works with Kwokwai Chan and Conan Leung on Hitchin connections in geometric quantization and deformation quantization.

- Geometric quantization and Hitchin connection
- Deformation quantization and algebraic index theorem
- Kähler quantization via Kapranov's L_∞ structure
- Quantum Master Equation and Hitchin connection.

Mathematical thoery of quantization

The phase space of a classical mechanical system is described geometrically as a symplectic manifold. The mathematical formulation of quantum mechanics is in terms of operators on Hilbert spaces satisfying the Dirac axioms. There are two schemes for quantization on symplectic manifolds: one is deformation quantization which focuses on the algebra of operators, and the other is geometric quantization which focuses on the Hilbert space of the quantum system.

The construction of geometric quantization (Hilbert spaces) depends on a choice of polarization. There are two types of polarizations: real and complex polarizations. We will focus on complex polarizations, which is equivalent to a complex structure on the phase space making it a Kähler manifold.

Geometric quantization and Toeplitz operators

On a Kähler manifold X, the Hilbert space (of level k) of its geometric quantization is defined as

$$\mathcal{H}_k := H^0_{\bar\partial}(X, L^{\otimes k}).$$

Here L denotes the prequantum line bundle on X.

On these Hilbert spaces $\mathcal{H}_k$, we can define Toeplitz operators associated to a smooth function $f \in C^\infty(X)$ as $T_{f,k} = \pi \circ m_f$. Here m_f denotes the multiplication by a smooth function $f \in C^\infty(X)$, and π denotes the orthogonal projection from smooth sections to holomorphic sections.

Toeplitz operators and deformation quantization

For two functions f and g, the compositions $T_{f,k} \circ T_{g,k}$ has the following asymptotic property as $k \to \infty$:

$$T_{f,k} \circ T_{g,k} \sim T_{fg,k} + \sum_{i \geq 1} \left(\frac{1}{k}\right)^i \cdot T_{C_i(f,g),k}.$$

This gives rise to a *deformation quantization* of smooth functions on X by turning $1/k$ into the formal variable $\hbar$.

Deformation quantization

Deformation quantization is an associative but non-commutative deformation of the algebra of smooth functions on symplectic manifolds.

Definition

Let (M, ω) be a symplectic manifold, then a deformation quantization of M is an associative product $*$ on $C^\infty(M)[[\hbar]]$ such that

$$f * g = f \cdot g + \sum_{i \geq 1} \hbar^i C_i(f, g),$$

where C_i's are bi-differential operators, with $C_1(f, g) - C_1(g, f) = \{f, g\}$.

Hitchin connections

Since this construction depends on the additional structure of polarization, a natural question arises from this choice: How to describe the independence of geometric quantization on this choice?

This question can be geometrically formulated as follows: Let $\mathcal{T}$ denote the parametrization space of a family of compatible complex polarizations (complex structures) on X. Then the Hilbert spaces $H^0(X_t, L^{\otimes k})$ for all $t \in \mathcal{T}$ form a vector bundle $\mathcal{H}_k$ over the parameter space $\mathcal{T}$. The standard differential geometry tells us that a connection on $\mathcal{H}_k$ induces parallel transport between fibers. However, in order to identify the fibers of $\mathcal{H}_k$ in a canonical way, we need a (projectively) flat connection.

To understand the role of a flat connection, we may compare it with the de Rham differential which detects constant functions.

Hitchin connections

The first and most important example of Hitchin connection is given independently by Hitchin and Axelrod-Della Pietra-Witten in Chern-Simons theory.

There has been studies on more examples of Hitchin connections. However, there was no systematic theory of Hitchin connections in general quantum theories, including its existence and properties.

The idea of my works is to use the theory of deformation quantization to give a systemic study of Hitchin connections.

Asymptotic Hitchin connection

As we study more on the quantization of symplectic (and Kähler) manifolds, we realize that we should require that Hitchin connections satisfy an asymptotic property with respect to the level k of the Hilbert space. (Recall that $\mathcal{H}_k = H^0(X, L^{\otimes k})$.) There are two reasons for this consideration:

- The Hitchin connections for Hilbert spaces of different levels $k > 0$ should be related, instead of being totally independent.
- Hitchin connection is a notion for both quantum observables and Hilbert spaces. And quantum observables are described in asymptotics as $k \to \infty$.

Batalin-Vilkovisky Quantization

The mathematical difficulty of quantum field theory involves the infinite dimensional path integral. The Batalin-Vilkovisky ("BV") formalism arose in the end of 1970's as a tool of mathematical physics designed to define and compute the path integral for gauge theories. In this formalism, the key ingredient is the Quantum Master Equation describing the quantum gauge consistency. There is a mathematical model for the path integral using homological algebra and Feynman graph computation:

$$\int_{\mathcal{E}} e^{S/\hbar},$$

where $\mathcal{E}$ is the space of fields and S the actions functional.

Motivation

In a joint work with Ryan Grady and Si Li, we constructed a sigma model with target a symplectic manifold X and considered its quantization in BV formalism. Using the technique of compactification of configuration spaces, we can show the following theorem:

Theorem (Grady-L-Li)

- *Quantum master equation of this sigma model is equivalent to the flatness of the Fedosov connection.*
- *The associative algebra induced from the factorization algebra of quantum observables is isomorphic to the Fedosov deformation quantization on the symplectic manifold X.*

Correlation function

Using the homological description of path integral in BV formalism, we can compute the correlation function of quantum observables (isomorphic to $C^\infty(X)[[\hbar]]$) as the following composition:

$$\langle - \rangle : Obs^q \cong C^\infty(X)[[\hbar]] \xrightarrow{\int_{Ber}} H^*_{dR}(M)[[\hbar]] \xrightarrow{\int_M} \mathbb{C}[[\hbar]].$$

This correlation function $\langle - \rangle$ satisfies two properties:

- The leading term of the correlation is the integral of the function f over M:

$$\langle f \rangle = \int_M f\omega^n + O(\hbar)$$

- The correlation of a star product is independent of the ordering.

$$\langle f * g \rangle = \langle g * f \rangle.$$

Trace

These two properties make the correlation function a *trace* on the algebra of quantum observables. In particular, the trace of the constant function 1 is called the *partition function*. By an explicit Feynman graph computation, we obtain a proof of the following *algebraic index* theorem:

Theorem (Grady-L-Li)

The partition function of the one-dimensional Chern-Simons model is given by the explicit formula:

$$\langle 1 \rangle = \int_M e^{-\omega/\hbar} \hat{A}_M$$

Kähler quantization

These results are still not completely satisfactory since we only see in these works the quantum observable which describes the algebraic structure of operators, but the Hilbert space is absent. The mathematical theory of Hilbert spaces in quantization is known as geometric quantization. For this, we also need the data of polarization.

We will now focus on the case of Kähler manifolds (i.e., symplectic manifold with compatible complex polarization). The goal is to give a quantization scheme in the BV quantization framework which encodes both the quantum observable and Hilbert spaces.

Quantization of the L_∞ structure

In 1996, Kapranov discovered an L_∞ structure on Kähler manifolds in his study of Rozansky-Witten theory. The main ingredient of this strucutre is a cochain complex whose cohomology is isomorphic to holomorphic functions, which gives an equivalent description of the complex structure (i.e., complex polarization) on X.

Theorem (Chan-Leung-L)

There exists a Fedosov connection D on X which is a quantum extension of Kapranov's L_∞ structure on Kähler manifolds.

According to the first results, this Fedosov connection induces a BV quantization of the Kähler manifold X. This quantization encodes not only the symplectic geometry (i.e., phase space) but also the complex geometry.

Fedosov connection on Kähler manifolds

We briefly describe the above Fedosov connection, i.e., quantum master equation. The idea behind Fedosov's original approach to deformation quantization is as follows: on the Weyl bundle $\mathcal{W}_{X,\mathbb{C}} := \widehat{\mathrm{Sym}}\, TX_{\mathbb{C}}^*$ over the phase space X, there exists a fiberwise star product $\star$ which describes the local picture of quantization on $\mathbb{C}^n$, and a (flat) Fedosov connection gives the gluing data for these local quantizations which is of the following form:

$$\nabla + \frac{1}{\hbar}[\gamma, -]_\star.$$

Here $[-,-]_\star$ denotes the bracket associated to the Wick product. The flat sections under this connections are isomorphic to the quantum observables $C^\infty(X)[[\hbar]]$.

We follow the same line of thought and give a construction of Bargmann-Fock bundle on Kähler manifolds, starting with the fiberwise Bargmann-Fock action of the Weyl bundle $\mathcal{W}_{X,\mathbb{C}}$ on $\mathcal{W}_X = \widehat{\mathrm{Sym}}\, TX^*$, which we denote by $\circledast$.

By twisting $\mathcal{W}_X$ with the tensor power of the prequantum line bundle $L^{1/\hbar}$, we obtain the Fock bundle $\mathcal{F}_X$ as a module sheaf over the Weyl bundle with respect to the fiberwise star product. There exists a Fedosov connection (i.e., quantum master equation) on the Fock bundle of the following form:

$$D = \nabla + \frac{1}{\hbar}\gamma \circledast -.$$

Bargmann-Fock bundle

Theorem (Chan-Leung-L)

- *The Fedosov connections on the Weyl bundle and Fock bundle are compatible in the sense that* $D(\alpha \circledast s) = D(\alpha) \circledast s + \alpha \circledast D(s)$.
- *Taking the evaluation* $\hbar = 1/k$, *the flat sections of the Fock bundle is isomorphic to* $H^0_{\bar{\partial}}(X, L^{\otimes k})$.

This result implies that we obtain a quantization scheme which involves both quantum observables and geometric quantization (Hilbert spaces) as modules, using the language of formal geometry and homological algebra.

Families of complex structures

This novel understanding of quantization can be applied to understanding Hitchin connections. First of all, the geometry of Weyl and Fock bundles can be constructed in families: Let $\mathcal{T}$ denote the space parametrizing a family of complex structures compatible with the symplectic form on X, we can simply take the pullback

$$\pi^*(\mathcal{W}_{X,\mathbb{C}}).$$

Here $\pi : X \times \mathcal{T} \to X$ denotes the projection. We can construct the Fock bundle in this geometric setting.

Quantum master equation in families

A Fedosov connection in the family case should involve not only those D_t's for each fixed complex structure $t \in \mathcal{T}$, but also an operator in the $\mathcal{T}$ direction which describes the variation of complex structures. The reason we need fiberwise Fedosov connection D_t is that the associated cohomology gives rise to the Hilbert spaces $H^0(X, L^{\otimes k})$ (after the evaluation $k = 1/\hbar$).

Thus we will consider those flat Fedosov connections on Weyl bundles in families (i.e., quantum master equations), which are of the following form:

$$d_{\mathcal{T}} + A + D_t$$

that is flat, i.e., squares to 0. Here $d_{\mathcal{T}}$ denotes de Rham differential on the parameter space $\mathcal{T}$ of complex structures, and A denotes a differential operator on the Weyl (or Fock) bundle valued in $\mathcal{A}^1(\mathcal{T})$.

Quantum master equation in families

Theorem (Chan-Leung-L)

- *An asymptotic Hitchin connection is equivalent to a quantum master equation in family Fock bundles on $X \times \mathcal{T}$.*
- *Quantum master equation in family Weyl bundle and Fock bundle exists simultaneously.*

In this way, the Hitchin connections we obtained are both on Hilbert spaces and quantum observables in a compatible way.

Quantum master equation in families

There will be several steps in the construction of a flat Fedosov connection in families. The first step is that there exists an operator $\nabla_{\mathcal{T}} = d_{\mathcal{T}} + A_1$ which can be defined on both Weyl and Fock bundles satisfying the following properties:

1. $\nabla_{\mathcal{T}}^2 = 0$.
2. $\nabla_{\mathcal{T}}$ is compatible with both the Wick products and the Bargmann-Fock action in the obvious sense.

The naive idea is to consider the operator $\nabla_{\mathcal{T}} + D_t$. However, the $\mathcal{A}_X^1 \wedge \mathcal{A}_{\mathcal{T}}^1$ component of the curvature $(\nabla_{\mathcal{T}} + D_t)^2$ does not vanish in general.

Quantum master equation in families

The geometric meaning of the vanishing of this component is that the parallel transport along the $\mathcal{T}$ direction preserves the Hilbert spaces $H^0(X_t, L^{\otimes k})$ (which is the cohomology of D_t) for different $t \in \mathcal{T}$. In this way, we obtain asymptotic Hitchin connections for both the Hilbert spaces and quantum observables.

For the vanishing of the component, the idea is to make certain corrections to $\nabla_{\mathcal{T}}$. We can show that this correction is possible if the family $\mathcal{T}$ of complex structures on X satisfies higher rigidity condition and condition that the cohomology class of the Ricci forms are propotional to that of the symplectic form.

Projective flatness and Maurer-Cartan equation

The last step is to look at the $\mathcal{A}^2_{\mathcal{T}}$ component of the curvature, which is equivalent to the (projective) flatness of the associated Hitchin connection.

Theorem (Chan-Leung-L)

Suppose the family of complex structures satisfies the above geometric conditions. Then there exists a Fedosov connection $\nabla_{\mathcal{T}} + A_2 + D_t$ if the Maurer-Cartan equation

$$\nabla_{\mathcal{T}} A_2 + A_2 \star_t A_2 = 0.$$

In particular, we obtain a series of cohomological obstructions to the existence of (projective) flat asymptotic Hitchin connections.

Thank You!