Lattice gauge theory and the discretization of Dirac operators

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Based on a joint work with

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Thm (FFMO'24)

- For $0 < a \ll 1$, we have

the index of a twisted Dirac op D on $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

$\parallel$

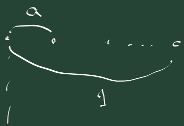
the index of the Wilson–Dirac op D_W on $\hat{\mathbb{T}}^d = (a\mathbb{Z}/\mathbb{Z})^d$

- The same holds for Clifford index, family index, and equivariant index.

My motivation comes from

- Seiberg-Witten theory
- DIFF = PL

in dimension 4



$= \hat{\mathbb{T}}^d$

§1 Index, spectral flow, and K-theory

For simply. d : even

$\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$

$$\begin{array}{c} E \\ \downarrow \\ \mathbb{T}^d \end{array}$$

hermitian

$\mathbb{Z}/2$-graded Clifford module bdl

$\hookrightarrow$ $\exists\ \gamma : E \to E$

$\exists\ c_j : E \to E \quad (j = 1, \cdots d)$

s.t. $\gamma = \gamma^*$ $\qquad$ $c_j = -c_j^*$

$\gamma^2 = \mathrm{id}$ $\qquad$ $c_j^2 = -\mathrm{id}$

$c_i c_j + c_j c_i = 0$ if $i \neq j$

$\gamma c_j + c_j \gamma = 0$

$\exists\ \gamma : E \to E$

$\exists\ c_j : E \to E \quad (j = 1, \dots, d)$

s.t. $\gamma = \gamma^*$ $\qquad c_j = -c_j^*$

$\gamma^2 = \mathrm{id}$ $\qquad c_j^2 = -\mathrm{id}$

$c_i c_j + c_j c_i = 0$ if $i \neq j$

$\gamma c_j + c_j \gamma = 0$

Fix a connection ∇ on E.

Dirac op $\quad D : \Gamma(E) \to \Gamma(E)$

$$D\Phi := \sum_{j=1}^{d} c_j \nabla_j \Phi$$

Ex $\quad d = 2$

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$= 2(\bar\partial)^*$

$$\rightsquigarrow \quad D = c_1 \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial y} = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & 0 \end{pmatrix}$$

$= 2\bar\partial$

Dirac op $\quad D : \Gamma(E) \to \Gamma(E)$

$$D\Phi := \sum_{j=1}^{d} c_j \nabla_j \Phi$$

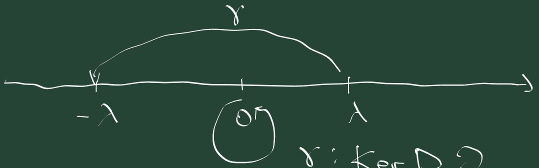
D is <u>self-adjoint</u> and <u>odd</u>

$$D = D^* \qquad\qquad D\gamma + \gamma D = 0$$

If $D\Phi = \lambda \Phi$,

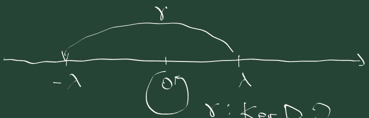
$$D(\gamma\Phi) = -\gamma D\Phi = -\gamma(\lambda\Phi) = (-\lambda)(\gamma\Phi)$$

$\rightsquigarrow$ Spec(D) is symmetric around the origin



$\gamma : \operatorname{Ker} D \circlearrowleft$

finite dim



$\gamma : \ker D$?

finite dim

Def.

$$\mathrm{ind}(D) := \mathrm{tr}(\gamma \mid \ker D)$$

$$= \dim\{\Phi \mid D\Phi = 0 \text{ and } \gamma\Phi = \Phi\}$$

$$\quad - \dim\{\Phi \mid D\Phi = 0 \text{ and } \gamma\Phi = -\Phi\}$$

Def.

$$\mathrm{ind}(D) := \mathrm{tr}(\gamma \mid \ker D)$$

$$= \dim\{\Phi \mid D\Phi = 0 \text{ and } \gamma\Phi = \Phi\}$$

$$- \dim\{\Phi \mid D\Phi = 0 \text{ and } \gamma\Phi = -\Phi\}$$

Next we consider a family

$$D + m\gamma$$

for $m \in [-1, 1]$

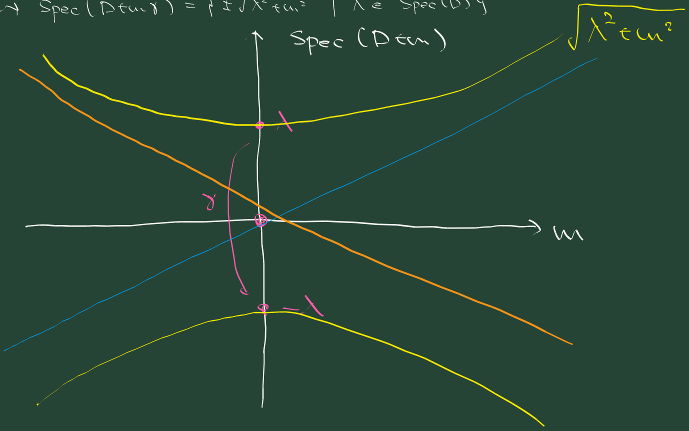
$$(D+m\gamma)^2 = D^2 + m(\overbrace{D\gamma + \gamma D}^{0}) + m^2 \overbrace{\gamma^2}^{\mathrm{id}}$$

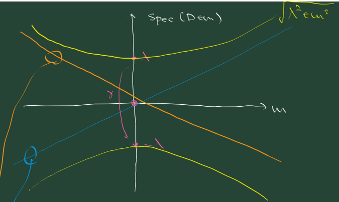
$$= D^2 + m^2$$

$\rightsquigarrow \mathrm{Spec}(D+m\gamma) = \{\pm\sqrt{\lambda^2+m^2} \mid \lambda \in \mathrm{Spec}(D)\}$

$$(D + m\gamma)^2 = D^2 + m(D\gamma + \gamma D) + m^2\gamma^2$$
$$= D^2 + m^2$$

$\rightsquigarrow$ $\mathrm{Spec}(D + m\gamma) = \{ \pm\sqrt{\lambda^2 + m^2} \mid \lambda \in \mathrm{Spec}(D) \}$





Def.

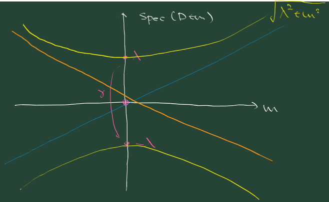
$$\begin{aligned} \mathrm{ind}(D) &:= \mathrm{tr}(\gamma \mid \ker D) \\ &= \boxed{\dim\{\Phi \mid D\Phi = 0 \text{ and } \gamma\Phi = \Phi\}} \\ &\quad - \boxed{\dim\{\Phi \mid D\Phi = 0 \text{ and } \gamma\Phi = -\Phi\}} \end{aligned}$$

If $D\Phi = 0$ and $\gamma\Phi = \pm\Phi$,

$$(D + m\gamma)\Phi = \pm m\Phi.$$

multiplicity $= \dim\{\Phi \mid D\Phi = 0 \text{ and } \gamma\Phi = +\Phi\}$

multiplicity $= \dim\{\Phi \mid D\Phi = 0 \text{ and } \gamma\Phi = -\Phi\}$



$$\boxed{\text{multiplicity} - \text{multiplicity}} = \mathrm{ind}(D)$$

$$\|$$

$$Sf(\{D+m\gamma\}_{m\in[-1,1]})$$

spectral flow.

Rmk $K^0(pt) \cong K^1([-1,1], \{\pm 1\})$

$$\quad\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\mathrm{ind}(D) = Sf(\{D+m\gamma\}_{m\in[-1,1]})$$

§ Dirac op on the lattice

Fix $N \in \{1, 2, \cdots\}$. $a := \frac{1}{N}$

$$\hat{\mathbb{T}}^d := (a\mathbb{Z}/\mathbb{Z})^d$$

$$\hat{E} = E|_{\hat{\mathbb{T}}^d}$$

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \in j$$

$$\downarrow$$

$$\hat{\mathbb{T}}^d$$

Using the parallel transport of ∇ on E,

$$\begin{cases} \hat{\nabla}_j^f \Phi(x) := \dfrac{\Phi(x + ae_j) - \Phi(x)}{a} & \text{for } x \in \hat{\mathbb{T}}^d \\ \hat{\nabla}_j^b \Phi(x) := \dfrac{\Phi(x) - \Phi(x - ae_j)}{a} \end{cases}$$

$$\begin{cases} \hat{\nabla}^f_j \Phi(x) := \dfrac{\Phi(x+ae_j) - \Phi(x)}{a} & \text{for } x \in \hat{\mathbb{T}}^d \\ \hat{\nabla}^b_j \Phi(x) := \dfrac{\Phi(x) - \Phi(x-ae_j)}{a} \end{cases}$$

$$\rightsquigarrow \quad (\hat{\nabla}^f_j)^* = -(\hat{\nabla}^b_j)$$

$\dim(\Gamma(\hat{E})) < \infty$

$\gamma : \underbrace{E^+}_{\gamma=+1} \rightleftarrows \underbrace{E^-}_{\gamma=-1}$

Lattice covariant derivative | naive Dirac op

$$\hat{\nabla}_j := \frac{\hat{\nabla}^f_j + \hat{\nabla}^b_j}{2}$$

$$\hat{D} := \sum c_j \hat{\nabla}_j$$

$$\rightsquigarrow \quad (\hat{\nabla}_j)^* = -\hat{\nabla}_j$$

$$\rightsquigarrow \begin{cases} \hat{D}^* = \hat{D} \\ \hat{D}\gamma + \gamma\hat{D} = 0 \end{cases}$$

$$\rightsquigarrow \boxed{\operatorname{ind}(\hat{D}) = 0}$$

③ Wilson–Dirac operator

The Wilson term

$$W := \frac{a}{2}\,\gamma \sum_{j=1}^{d} (\hat{\nabla}_j^f)^* (\hat{\nabla}_j^f)$$

<u>Rmk</u> No continuous counterpart.

$\rightsquigarrow$ · $W^* = W$

· $W\gamma = \gamma W$ $\leftarrow$ W is <u>not</u> odd.

The Wilson term

$$W := \frac{a}{2} \gamma \sum_{j=1}^{d} (\hat{\nabla}_j^f)^* (\hat{\nabla}_j^f)$$

$$\hat{\nabla}_j^f \Phi(x) := \frac{\Phi(x + a e_j) - \Phi(x)}{a}$$

$$\hat{\nabla}_j := \frac{\hat{\nabla}_j^f + \hat{\nabla}_j^b}{2}$$

$$\hat{\nabla}_j^b \Phi(x) := \frac{\Phi(x) - \Phi(x - a e_j)}{a}$$

<u>naive</u> Dirac op

$$\hat{D} := \sum c_j \hat{\nabla}_j$$

<u>Def</u> Wilson-Dirac op

$$D_W := \hat{D} + W$$

$$\begin{cases} (D_W)^* = D_W \\ D_W \text{ is } \underline{\text{not}} \text{ odd.} \end{cases}$$

Def Wilson-Dirac op

$$D_W := \not{D} + W$$

$$\left\{ \begin{array}{l} (D_W)^* = D_W \\ D_W \text{ is } \underline{\text{not}} \text{ odd.} \end{array} \right.$$

Thus, it is not so obvious to define the index of D_W

$\rightsquigarrow$ spectral flow

Def

For $0 < a \ll 1$,

$$\mathrm{ind}(D_W) := \mathrm{sf}\left(\{D_W + m\gamma\}_{m \in [-1,1]}\right)$$

Thm (FFMOYY)

$\exists a_0 = \frac{1}{N_0}$ $\forall a = \frac{1}{N} \in (0, a_0)$

s.t. $D_W + m\gamma$ is invertible at $m = +1$ and $m = -1$

§ Main thm

$$\boxed{\begin{array}{l} E \\ \downarrow \\ \mathbb{T}^d \\[1em] D := \sum c_j \partial_j \\[1em] \operatorname{ind}(D) \end{array}}$$

$$\xrightarrow{a \ll 1}$$

$$\boxed{\begin{array}{l} \hat{E} = E|_{\hat{\mathbb{T}}^d} \\ \downarrow \\ \hat{\mathbb{T}}^d = (a\mathbb{Z}/\mathbb{Z}) \\[1em] \hat{D} := \sum c_j \hat{\nabla}_j \\[1em] W := \frac{a}{2}\gamma \sum (\hat{\nabla}_j)^* (\hat{\nabla}_j) \\[1em] D_W := \hat{D} + W \\[1em] \operatorname{ind}(D_W) := \operatorname{sf}(\{D_W + m\gamma\}) \end{array}}$$

$$K^0(pt) \xrightarrow{\cong} K^1([-1,1], \{\pm 1\})$$

$$\operatorname{ind}(D) = \operatorname{sf}(D + m\gamma)$$

$$\operatorname{ind}(D_W) := \operatorname{sf}(D_W + m\gamma)$$

$$\begin{array}{c} E \\ \downarrow \\ \mathbb{T}^d \end{array}$$

$D := \sum c_j \nabla_j$

$\mathrm{ind}(D)$

$\overset{a \ll 1}{\rightsquigarrow}$

$$\begin{array}{c} \hat{E} = E|_{\hat{\mathbb{T}}^d} \\ \downarrow \\ \hat{\mathbb{T}}^d = (a\mathbb{Z}/\mathbb{Z}) \end{array}$$

$\hat{D} := \sum c_j \hat{\nabla}_j$

$W := \frac{a}{2} \gamma \sum (\hat{\nabla}_j)^* (\hat{\nabla}_j)$

$D_W := \hat{D} + W$

$\mathrm{ind}(D_W) := \mathrm{sf}(\{D_W + m\gamma\})$

$$\begin{array}{ccc} K^0(pt) & \xrightarrow{\cong} & K^1([-1,1], \{\pm 1\}) \\ \downarrow & & \downarrow \\ \mathrm{ind}(D) & = & \mathrm{sf}(D + m\gamma) \end{array}$$

$\mathrm{ind}(D_W) := \mathrm{sf}(D_W + m\gamma)$

Thm (FFMOYY)

$\exists\, a_0 = \frac{1}{N_0}$ $\forall\, a = \frac{1}{N} \in (0, a_0)$

$$\mathrm{ind}(D) = \mathrm{ind}(D_W)$$

The same holds for Clifford ind, family index, and equivariant index