The second law of information thermodynamics for general measurement processes

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arXiv: 2308.15558

1. Introduction

- Maxwell's demon[1]

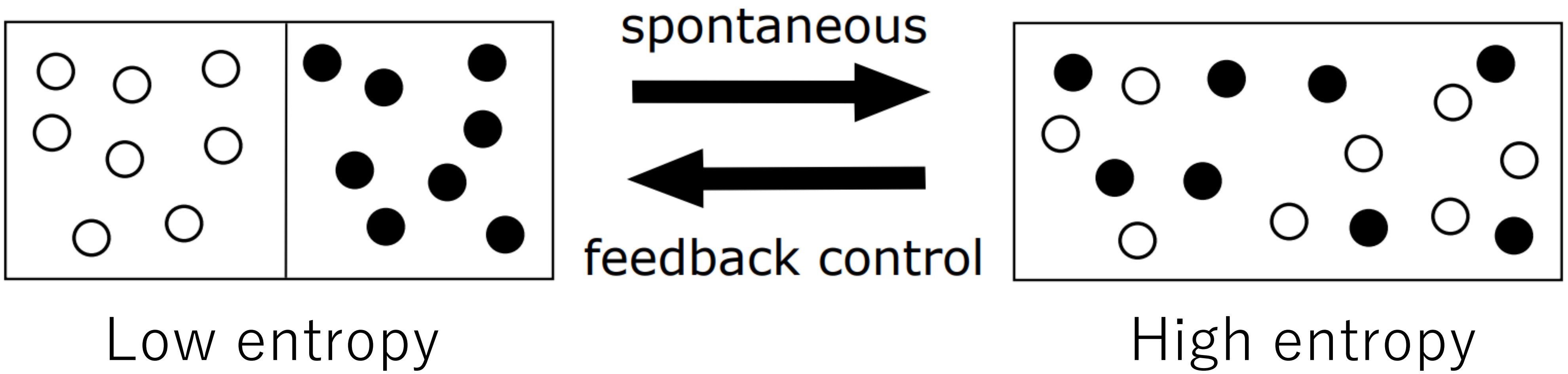

Fig. 1. Maxwell's demon[1]

Feedback control can decrease entropy
→ How to make feedback control and 2nd law consistent?

[1] J. C. Maxwell, Theory of heat (Appleton, London, 1871)

1. Introduction

- Sagawa and Ueda’s solution
  - Sagawa and Ueda (2008)[2]: feedback control allows us to extract more work beyond the conventional second law
  - Sagawa and Ueda (2009)[3, 4]: however, the work feedback controller needs for the measurement and resetting its memory (erasure) compensates the excess

[2] T. Sagawa and M. Ueda, Phys. Rev. Lett. 100, 080403 (2008)
[3] T. Sagawa and M. Ueda, Phys. Rev. Lett. 102, 250602 (2009)
[4] T. Sagawa and M. Ueda, Phys. Rev. Lett. 106, 189901 (2011) (the erratum of [3])

1. Introduction

Problem: The measurement process considered by Sagawa and Ueda is not general

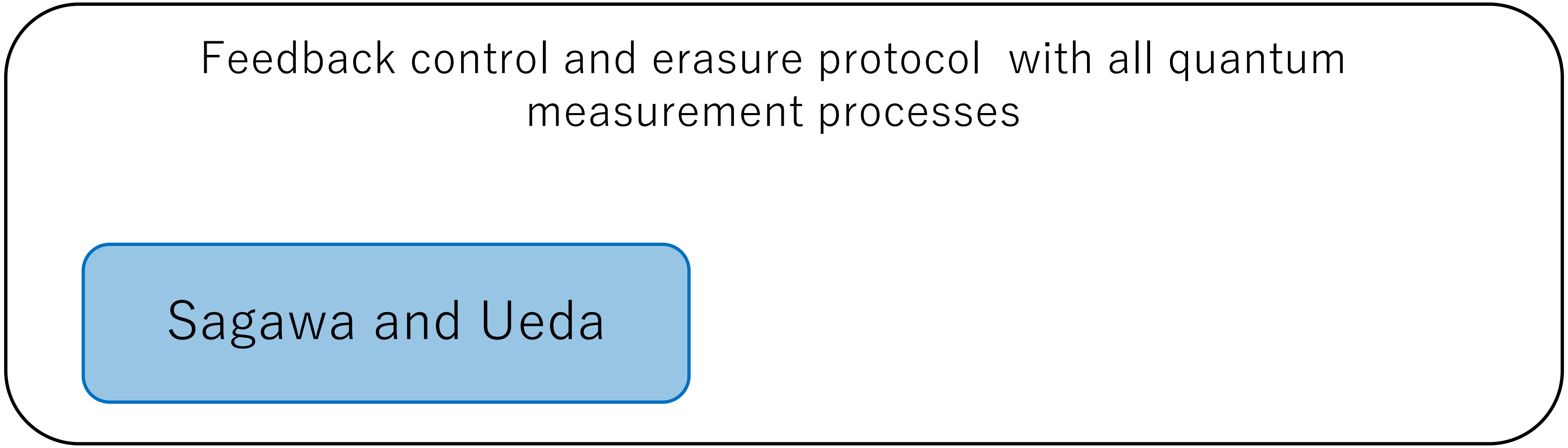

Fig. 2. Sagawa and Ueda and our work

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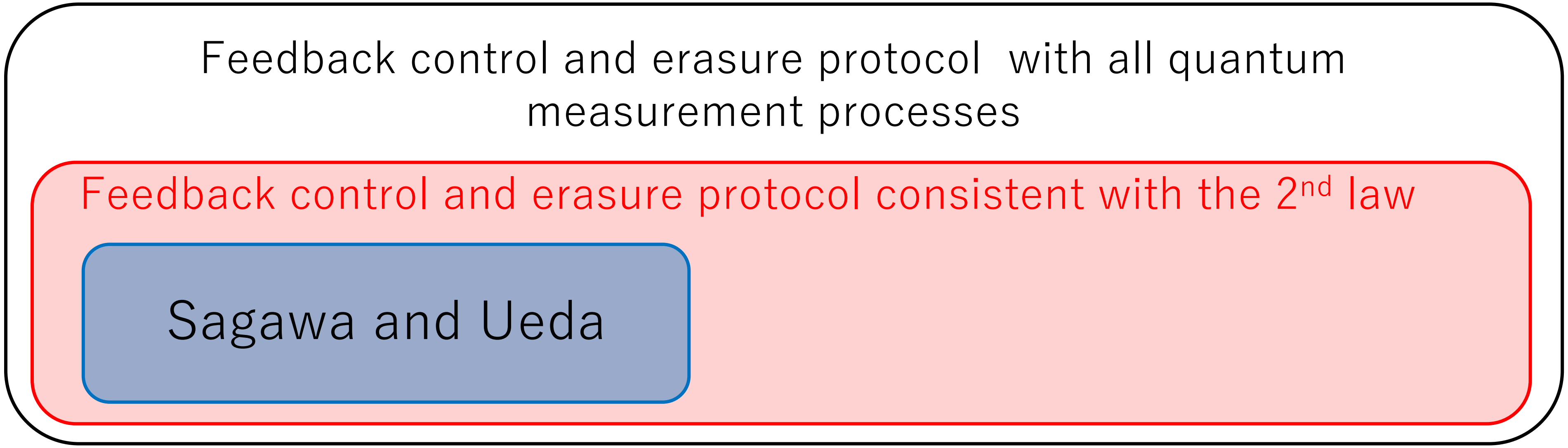

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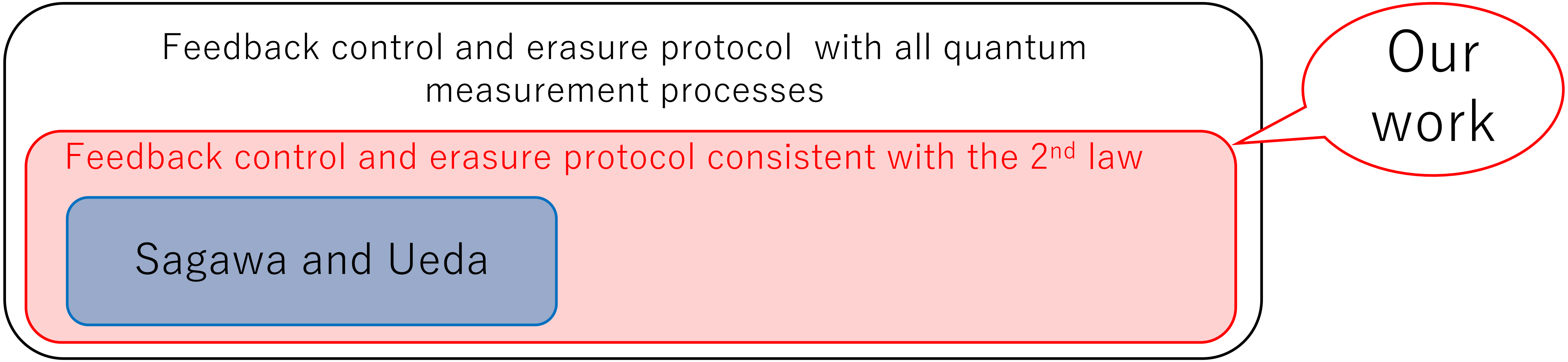

Fig. 2. Sagawa and Ueda and our work

2. Thermodynamics

- The first law of thermodynamics

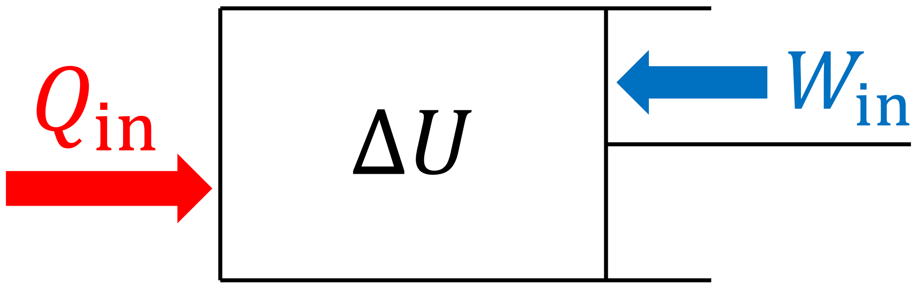

Fig. 3. Internal energy change due to heat and work

$$\Delta U = Q_{\mathrm{in}} + W_{\mathrm{in}}$$

2. Thermodynamics

- The second law of thermodynamics
  - Kelvin's principle: No cycle can convert all heat received from one heat bath to work
  - Clausius' principle: Heat does not spontaneously move from a low temperature object to a high temperature object
  - Entropy non-decreasing law (an isolated system or an adiabatic process): $\Delta S \geq 0$
  - $W_{\text{ext}} \leq -\Delta F$ (an isothermal process)

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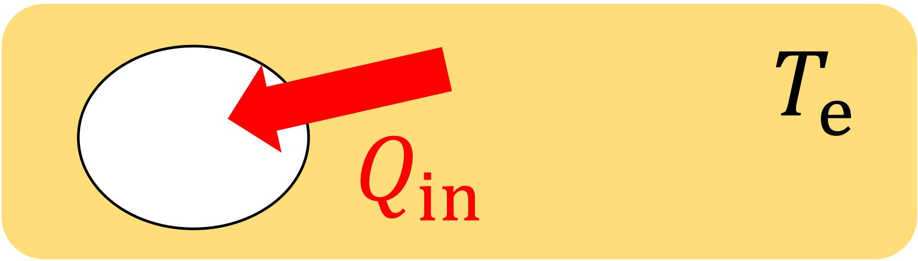


Fig. 4. Isothermal process

- Helmholtz free energy: $F \coloneqq U - TS$

Isothermal process: $\Delta F = \Delta U - T_{\mathrm{e}} \Delta S$

The 1st law: $\Delta U = W_{\mathrm{in}} + Q_{\mathrm{in}}$

Clausius' inequality: $T_{\mathrm{e}} \Delta S - Q_{\mathrm{in}} \geq 0$

$$W_{\mathrm{in}} \geq \Delta F$$

or

$$W_{\mathrm{ext}} \leq -\Delta F$$

(Clausius-type inequality)

3. Basic notions

- Hamiltonian: H^A
- Averaged energy: $E(\rho^A; H^A) \coloneqq \mathrm{Tr}[\rho^A H^A]$
- Inverse temperature: $\beta \coloneqq \frac{1}{kT} > 0$
- Partition function: $Z^A \coloneqq \mathrm{Tr}\, e^{-\beta H^A}$
- Gibbs (thermal equilibrium) state: $\gamma^A \coloneqq \frac{e^{-\beta H^A}}{Z^A}$
- Equilibrium free energy: $F_{\mathbf{eq}}^A \coloneqq -\beta^{-1} \ln Z^A$

3. Basic notions

- Von Neumann entropy[5]: $S(\rho^A)$ or $S(A)_\rho := -\mathrm{Tr}[\rho^A \ln \rho^A]$
- Umegaki relative entrpy[6]:

$$D(\rho^A || \sigma^A) \coloneqq \mathrm{Tr}[\rho^A \ln \rho^A - \rho^A \ln \sigma^A] \ (\sigma^A \geq 0)$$

- Quantum mutual information: $I(A\!:\!A')_\rho \coloneqq S(A)_\rho + S(A')_\rho - S(AA')_\rho$
- *Nonequilibrium free energy* [7]: $F(\rho^A; H^A) \coloneqq F_{\mathrm{eq}}^A + \beta^{-1} D(\rho^A || \gamma^A)$

Important formula[7] : $F(\rho^A; H^A) = E(\rho^A; H^A) - \beta^{-1} S(A)_\rho$

[5] J. von Neumann, Mathematical foundations of quantum mechanics (Princeton University Press, 1955).
[6] H. Umegaki, Proc. Japan Acad. 37, 459 (1961).
[7] M. Esposito and C. Van den Broeck, EPL (Europhysics Letters) 95, 40004 (2011).

3. Basic notions

- Quantum instruments and indirect measurement [8]

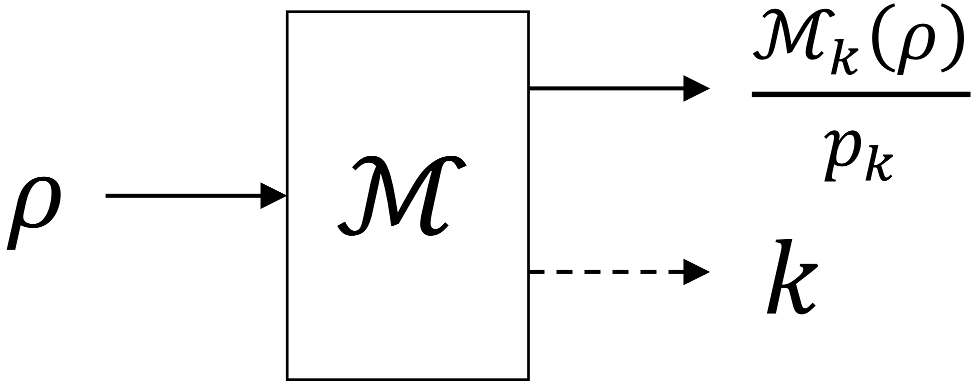


Fig. 5 Quantum instrument

$\mathcal{M}_k$ is a CP linear map and $\sum_k \mathcal{M}_k$ is TP

[8] M. Ozawa, J. Math. Phys. 25, 79 (1984).

3. Basic notions

- We say that instrument $\mathcal{M}$ is *efficient* if $\mathcal{M}_k$ has only one Kraus operator for each measurement outcome k
- *Groenewold--Ozawa information gain* [9, 10] is defined as

$$I_{\mathrm{GO}} := S(A)_{\mathrm{pre}} - S(A|K)_{\mathrm{post}}$$

- QC-mutual information[2] is defined as
  - $I_{QC} \coloneqq S(A)_{\mathrm{pre}} - \sum_k p_k S\left(\frac{\sqrt{E_k}\rho\sqrt{E_k}}{p_k}\right)$ $(E_k\!: \mathrm{POVM})$

For efficient instruments, $I_{\mathrm{GO}} = I_{\mathrm{QC}}$[11]

[9] H. J. Groenewold, Int. J. Theor. Phys. 4, 327 (1971).
[10] M. Ozawa, J. Math. Phys. 27, 759 (1986).
[11] F. Buscemi, M. Hayashi, and M. Horodecki, Phys. Rev. Lett. 100, 210504 (2008).

4. Work for controller-assisted information processing

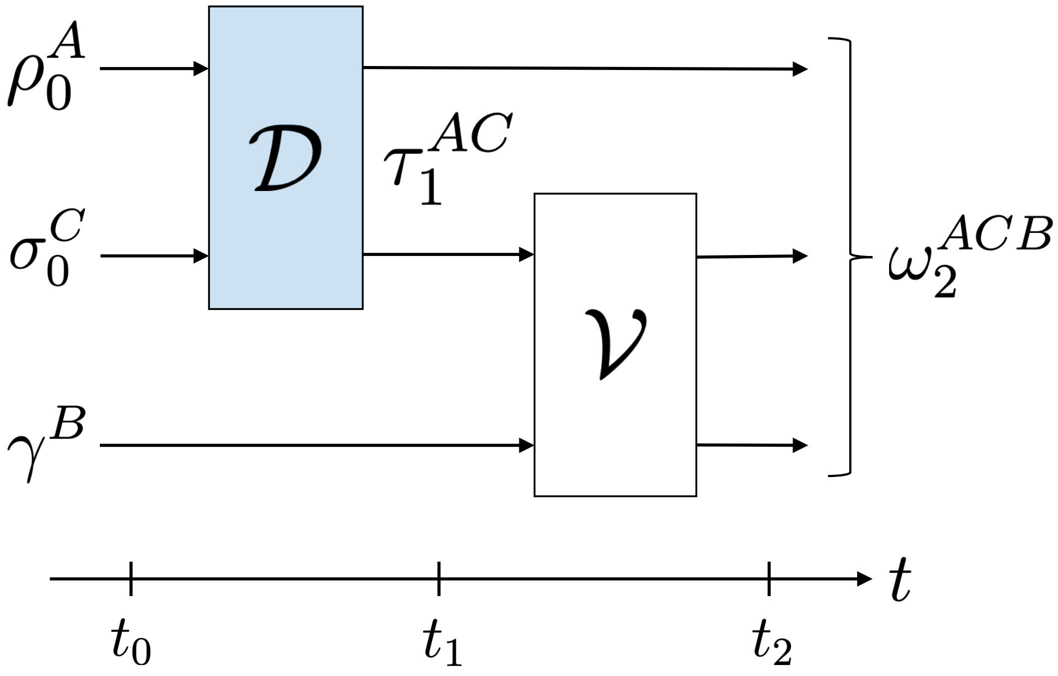


Fig. 7. controller-assisted information processing followed by a unitary interaction with the thermal bath

$$W_{\text{ext}}^A \coloneqq -\Delta E_{0\to 1}^A = E(\rho_0^A) - E(\tau_1^A)$$

The first law during 1→2

$$\Delta E_{1\to 2}^C = W_{\text{in},\mathcal{V}}^C + Q$$

where $Q \coloneqq -\Delta E_{1\to 2}^B$ is heat absorbed by the system

$$\rightarrow W_{\text{in},\mathcal{V}}^C = \Delta E_{1\to 2}^{CB}$$

S. M., K. Sakai, K. Kato, and F. Buscemi, arXiv: 2308.15558 (2023)

4. Work for controller-assisted information processing

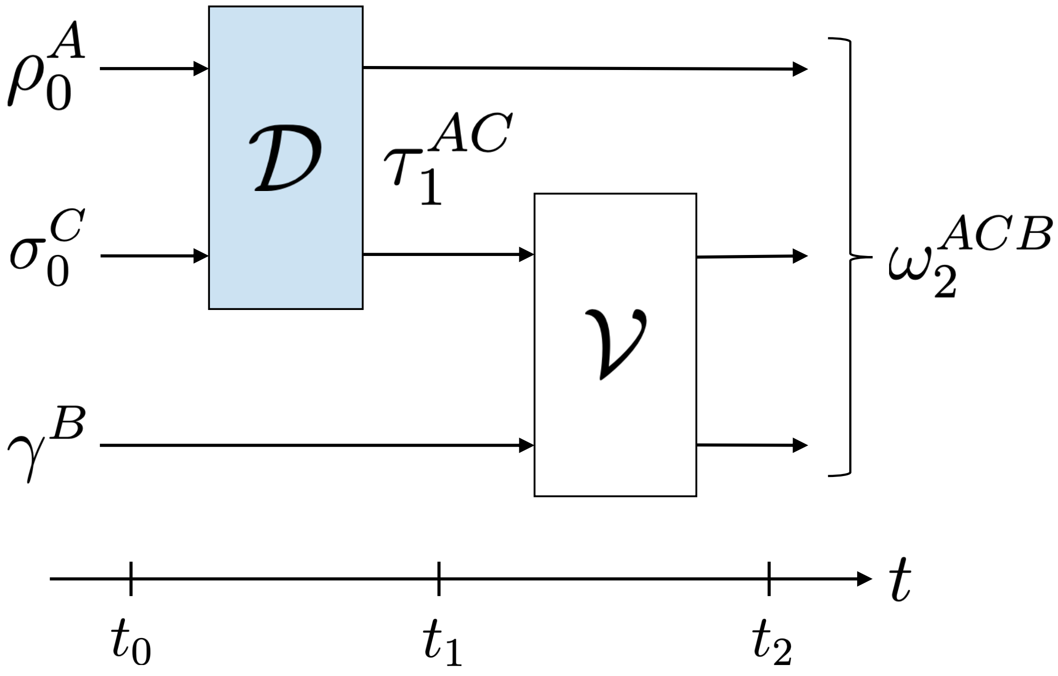


Fig. 7. controller-assisted information processing followed by a unitary interaction with the thermal bath

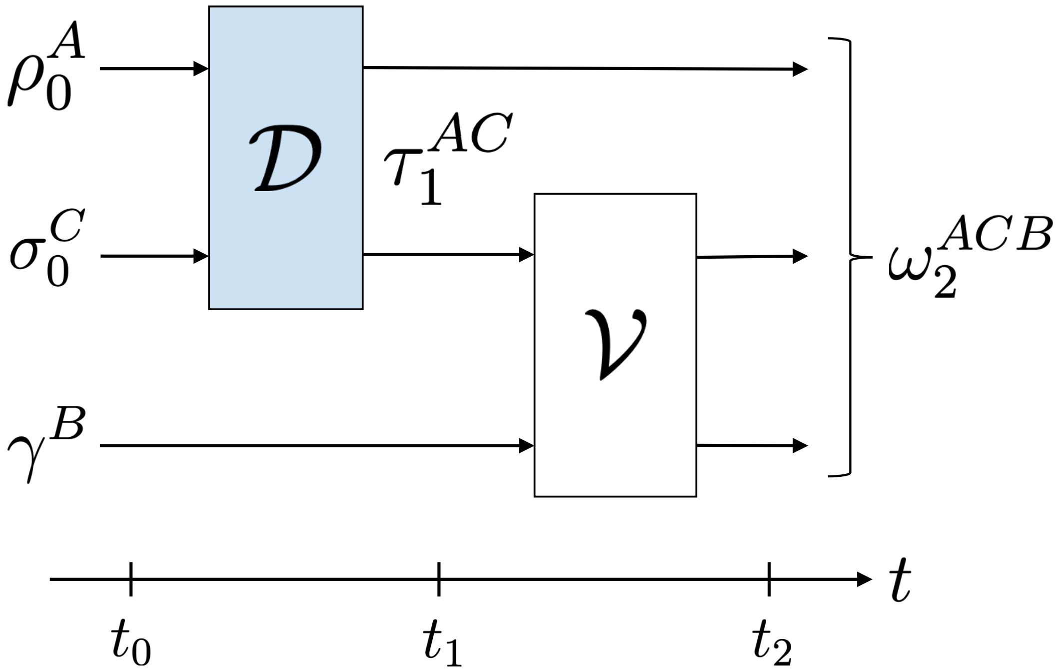
$$W_{\mathrm{ext}}^{A} := -\Delta E_{0\to 1}^{A} = E\left(\rho_0^A\right) - E\left(\tau_1^A\right)$$

$$W_{\mathrm{in},\mathcal{V}}^{C} = \Delta E_{1\to 2}^{CB}$$

$$W_{\mathrm{in}}^{C} := \Delta E_{0\to 1}^{C} + \Delta E_{1\to 2}^{CB}$$

$$W_{\mathrm{ext}}^{AC} := W_{\mathrm{ext}}^{A} - W_{\mathrm{in}}^{C}$$

4. Work for controller-assisted information processing



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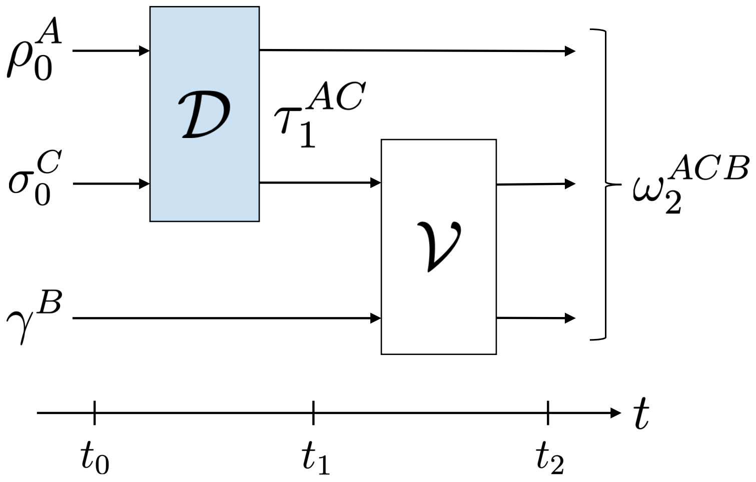

$$\Delta I(A:C)_{1\to 2} \coloneqq I(A:C)_2 - I(A:C)_1 \leq 0 \text{ [12]}$$
$$I(C:B)_{\omega_2} \geq 0,\ D(\omega_2^B \| \gamma^B) \geq 0 \text{[13]}$$

[12] B. Schumacher and M. A. Nielsen, Phys. Rev. A 54, 2629 (1996).
[13] O. Klein, Z. Phys. 72, 767 (1931).

$$W_{\text{ext}}^{AC} = -\Delta F_{0\to 2}^{AC} - \beta^{-1}[\Delta S_{0\to 1}^{AC} - \Delta I(A:C)_{1\to 2} + I(C:B)_{\omega_2} + D(\omega_2^B \| \gamma^B)]$$

Theorem 1 (Clausius-type inequality)

$$W_{\text{ext}}^{AC} \leq -\Delta F_{0\to 2}^{AC}$$
$$\Leftrightarrow \Delta S_{0\to 1}^{AC} - \Delta I(A:C)_{1\to 2} + I(C:B)_{\omega_2} + D(\omega_2^B \| \gamma^B) \geq 0$$

5. Feedback control and erasure protocol

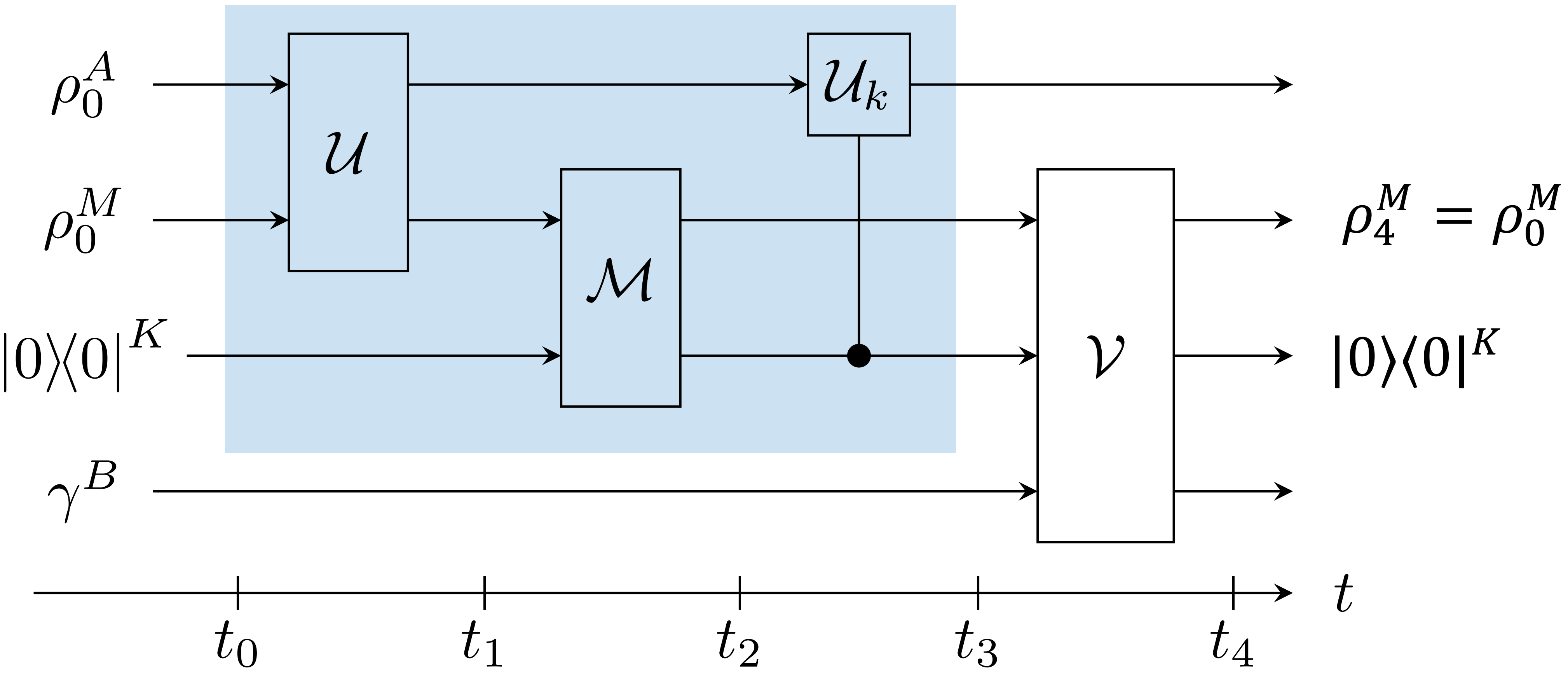

Fig. 8. Feedback control and erasure protocol

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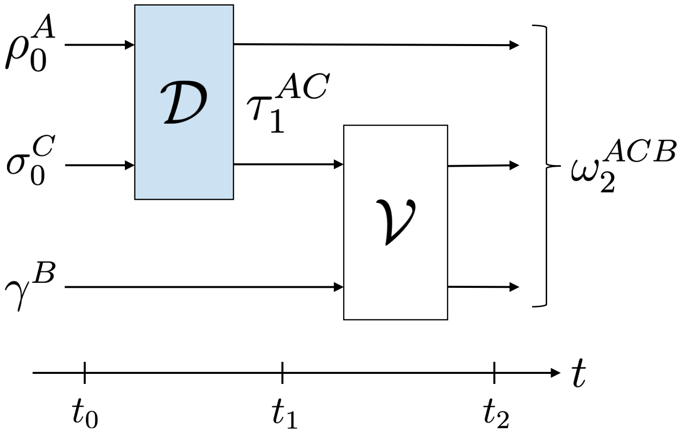
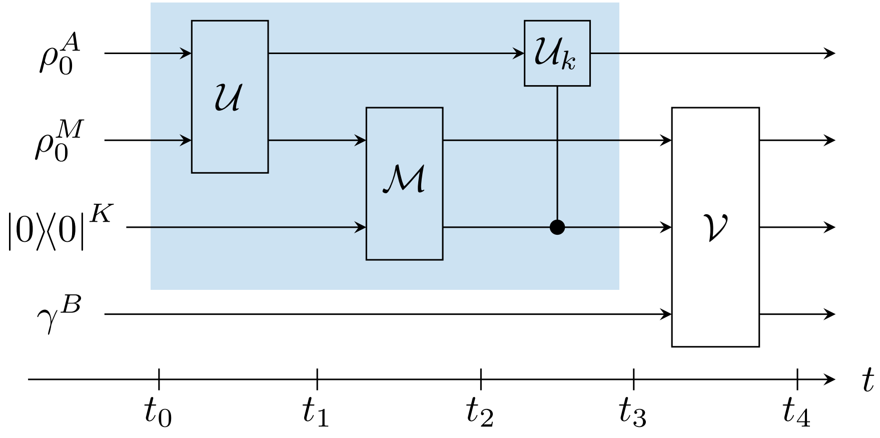
5. Feedback control and erasure protocol



$C \to MK$



Theorem 1

$$W_{\mathrm{ext}}^{AC} \leq -\Delta F_{0\to 2}^{AC}$$
$$\Leftrightarrow \Delta S_{0\to 1}^{AC} - \Delta I(A{:}C)_{1\to 2} + I(C{:}B)_{\omega_2} + D(\omega_2^B || \gamma^B) \geq 0$$

Theorem 2

$$W_{\mathrm{ext}}^{AMK} \leq -\Delta F_{0\to 4}^{AMK}$$
$$\Leftrightarrow \Delta S_{0\to 2}^{AMK} - \Delta I(A{:}MK)_{3\to 4} + I(MK{:}B)_{\rho_4} + D(\rho_4^B || \gamma^B) \geq 0$$

6. A generalized second law of information thermodynamics

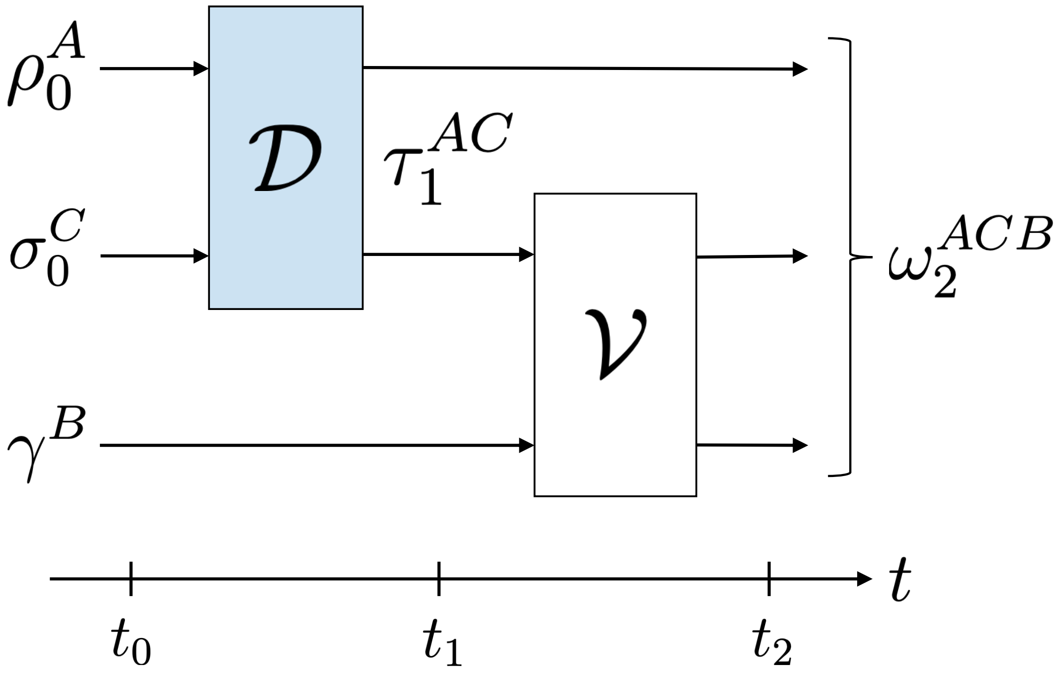


Fig. 7. controller-assisted information processing followed by a unitary interaction with the thermal bath

$$W_{\mathrm{ext}}^{A} := -\Delta E_{0\to 1}^{A} = E\left(\rho_0^A\right) - E\left(\tau_1^A\right)$$

$$W_{\mathrm{in},\mathcal{V}}^{C} = \Delta E_{1\to 2}^{CB}$$

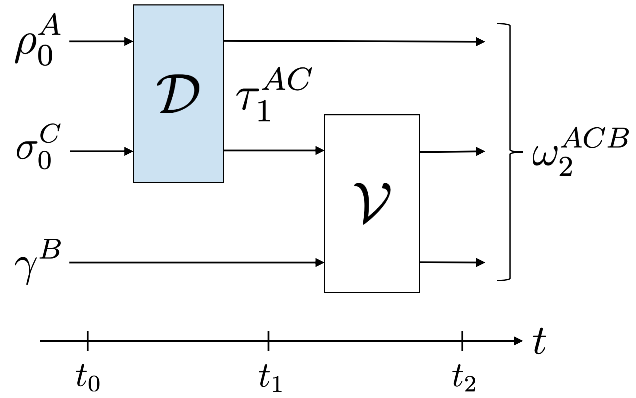
$$W_{\mathrm{in}}^{C} := \Delta E_{0\to 1}^{C} + \Delta E_{1\to 2}^{CB}$$

$$W_{\mathrm{ext}}^{AC} := W_{\mathrm{ext}}^{A} - W_{\mathrm{in}}^{C}$$

6. A generalized second law of information thermodynamics

Work formula for the work extractable from the system:

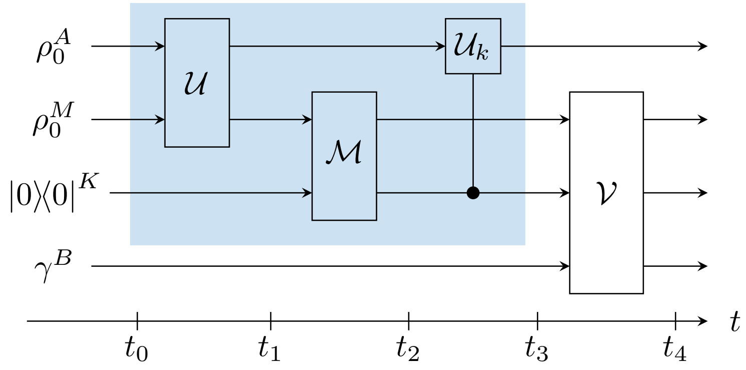
$$W_{\mathrm{ext}}^{A} = -\Delta F_{0\to 2}^{A} - \beta^{-1}\Delta S_{0\to 2}^{A}$$



- $\Delta F_{0\to 2}^{A} \to \Delta F_{0\to 4}^{A}, \Delta S_{0\to 2}^{A} \to \Delta S_{0\to 3}^{A}$
- $S(A)_{\rho_0} - S(A)_{\rho_3} = S(A)_{\rho_0} - S(A|K)_{\rho_2} - I(A;K)_{\rho_3}$
  $\leq S(A)_{\rho_0} - S(A|K)_{\rho_2} \quad (I(A;K)_{\rho_3} \geq 0[13])$

Theorem3 $\quad W_{\mathrm{ext}}^{A} \leq -\Delta F_{0\to 4}^{A} + \beta^{-1} I_{\mathrm{GO}}$

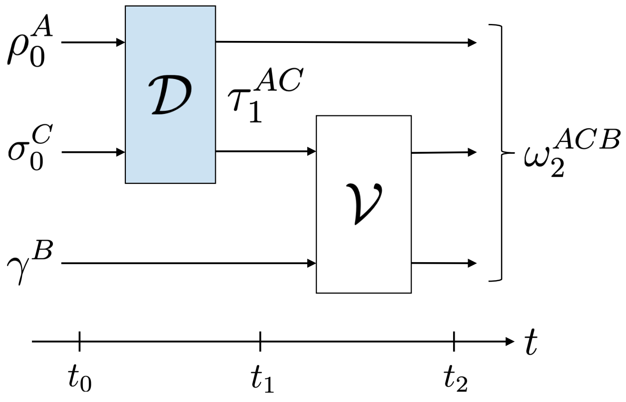
$(I_{\mathrm{GO}} \coloneqq S(A)_{\rho_0} - S(A|K)_{\rho_2})$



6. A generalized second law of information thermodynamics

Work formula of the controller:

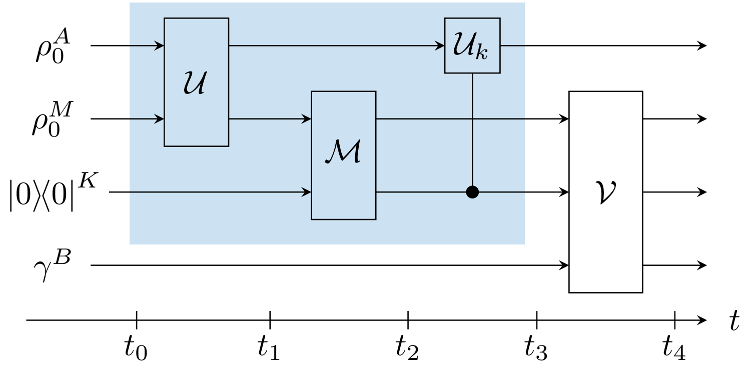
$$W_{\mathrm{in}}^{C} = \Delta F_{0\to 2}^{C} + \beta^{-1}[\Delta S_{0\to 1}^{C} + I(C{:}B)_{\omega_2} + D(\omega_2^B||\gamma^B)]$$



- $C \to MK$
- $\Delta S_{0\to 1}^{C} \to \Delta S_{0\to 2}^{MK}$
- $D(\omega_2^B||\gamma^B) \to D(\rho_4^B||\gamma^B) \geq 0$
- $\Delta F_{0\to 2}^{C} \to \Delta F_{0\to 4}^{MK} = 0$ (because of the erasure)
- Strong subadditivity of entropy[14]

Theorem4

$$W_{\mathrm{in}}^{MK} \geq \beta^{-1}[\Delta S_{0\to 2}^{AMK} + I_{\mathrm{GO}}]$$

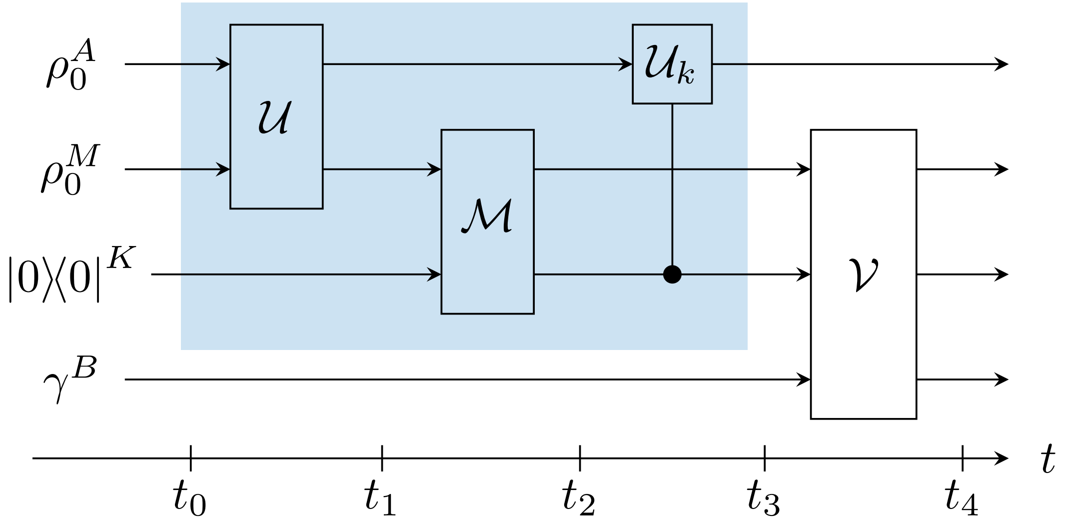


[14] E. H. Lieb and M. B. Ruskai, J. Math. Phys. 14, 1938 (2003)

6. A generalized second law of information thermodynamics

General measurement process

- $W_{\mathrm{ext}}^{A} \leq -\Delta F_{0\to 4}^{A} + \beta^{-1} I_{\mathrm{GO}}$
- $W_{\mathrm{in}}^{MK} \geq \beta^{-1}[\Delta S_{0\to 2}^{AMK} + I_{\mathrm{GO}}]$



If $\mathcal{M}$ is entropy non-reducing:

$$\Delta S_{0\to 2}^{AMK} \geq 0$$

we have

$$W_{\mathrm{in}}^{MK} \geq \beta^{-1} I_{\mathrm{GO}}$$

A generalized second law of information thermodynamics

$$W_{\mathrm{ext}}^{AMK} = W_{\mathrm{ext}}^{A} - W_{\mathrm{in}}^{MK} \leq -\Delta F_{0\to 4}^{A}$$

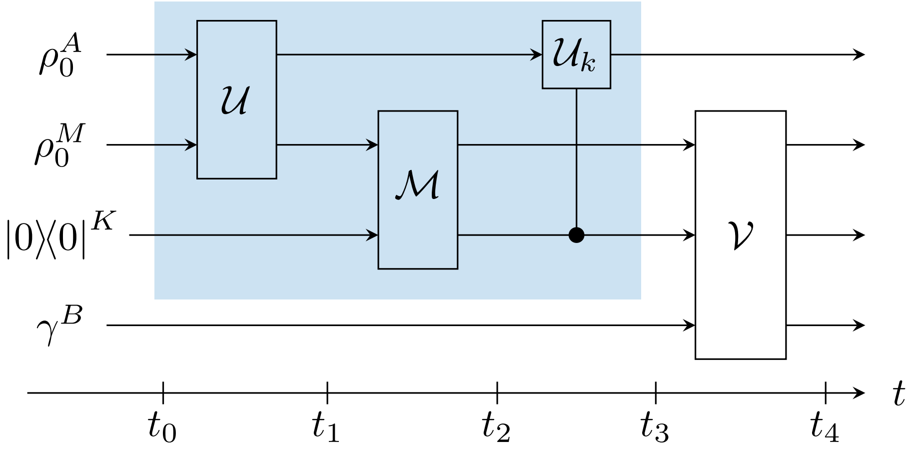
6. A generalized second law of information thermodynamics

Our result (valid for general measurement processes

- $W_{\mathrm{ext}}^{A} \leq -\Delta F_{0\to4}^{A} + \beta^{-1} I_{\mathrm{GO}}$
- $W_{\mathrm{in}}^{MK} \geq \beta^{-1}[\Delta S_{0\to2}^{AMK} + I_{\mathrm{GO}}]$

Sagawa and Ueda's results[2, 3, 4]

- $W_{\mathrm{ext}}^{A} \leq -\Delta F_{\mathrm{eq},0\to4}^{A} + \beta^{-1} I_{\mathrm{QC}}$
- $W_{\mathrm{in}}^{MK} \geq I_{\mathrm{QC}}$



6. A generalized second law of information thermodynamics

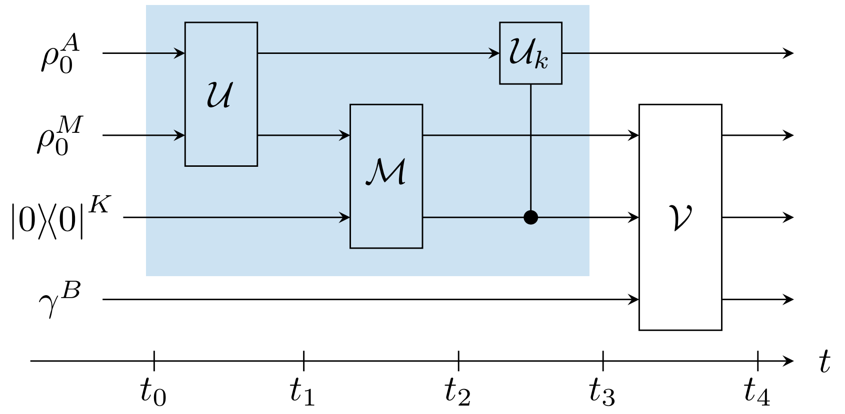
- Sagawa and Ueda's assumption[2, 3, 4]

  (A1) $\mathcal{M}$ is projection→ $\Delta S^{AMK}_{0\to 2} \geq 0$[15]

  (A2) The target system's measurement process is efficient→$I_{\mathrm{GO}} = I_{\mathrm{QC}}$ [11]

  (A3) The target system's initial state is the Gibbs state→$-\Delta F^{A}_{0\to 4} \leq -\Delta F^{A}_{\mathrm{eq},0\to 4}$

[15] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information: 10th Anniversary Edition (Cambridge University Press, 2010)



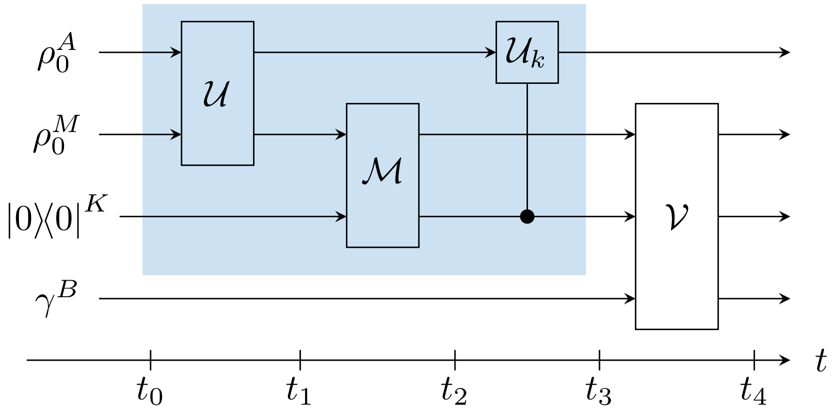
+

$$W^{A}_{\mathrm{ext}} \leq -\Delta F^{A}_{0\to 4} + \beta^{-1} I_{\mathrm{GO}}$$
$$W^{MK}_{\mathrm{in}} \geq \beta^{-1}[\Delta S^{AMK}_{0\to 2} + I_{\mathrm{GO}}]$$

Sagawa and Ueda's results

$$W^{A}_{\mathrm{ext}} \leq -\Delta F^{A}_{\mathrm{eq},0\to 4} + \beta^{-1} I_{\mathrm{QC}}$$
$$W^{MK}_{\mathrm{in}} \geq I_{\mathrm{QC}}$$

6. A generalized second law of information thermodynamics



- Abdelkhalek et al. (2016)'s assumption[16]

  (A1) $\mathcal{M}$ is projection→ $\Delta S_{0\to 2}^{AMK} \geq 0$

+

- $W_{\text{ext}}^A \leq -\Delta F_{0\to 4}^A + \beta^{-1} I_{\text{GO}}$
- $W_{\text{in}}^{MK} \geq \beta^{-1}[\Delta S_{0\to 2}^{AMK} + I_{\text{GO}}]$

⇒ $W_{\text{ext}}^A - W_{\text{in}}^{MK} \leq -\Delta F_{0\to 4}^A$

[16] K. Abdelkhalek, Y. Nakata, and D. Reeb, arXiv:1609.06981 (2016).

7. Conclusion

- We analyze the work associated with the feedback control with a general quantum measurement process
- Necessary and sufficient conditions for the Clausius-type inequality was obtained

$$W_{\mathrm{ext}}^{AMK} \leq -\Delta F_{0\to4}^{AMK} \Leftrightarrow \Delta S_{0\to2}^{AMK} - \Delta I(A{:}MK)_{3\to4} + I(MK{:}B)_{\rho_4} + D(\rho_4^B||\gamma^B) \geq 0$$

- Two bounds of work hold in general measurement processes:
  - $W_{\mathrm{ext}}^{A} \leq -\Delta F_{0\to4}^{A} + \beta^{-1} I_{\mathrm{GO}}$
  - $W_{\mathrm{in}}^{MK} \geq \beta^{-1}[\Delta S_{0\to2}^{AMK} + I_{\mathrm{GO}}]$
- Entropy non-reducing measurement→ $W_{\mathrm{ext}}^{A} - W_{\mathrm{in}}^{MK} \leq -\Delta F_{0\to4}^{A}$

S. M., K. Sakai, K. Kato, and F. Buscemi, arXiv: 2308.15558 (2023)