A review of vertex algebras and chiral algebras

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§. Introduction

- Vertex algebras were introduced by Borcherds (1986), and they are known as an algebraic framework of two-dimensional conformal field theory (CFT).
- On the other hand, there is a geometric framework of CFT, which is called a chiral algebra. This notion was introduced by Beilinson and Drinfeld (2004) using D-modules and operads.

(1) $\left.\begin{array}{l} X : \text{smooth curve} \\ \mathcal{V} : \text{(quasi) CVA} \end{array}\right\} \rightsquigarrow \mathcal{V}_X^r$: chiral algebra on X.

(2) The chiral operad P^{ch} and its algebraic counterpart.

§. Vertex Algebras.

- V : $\mathbb{C}$-linear space (space of state)
- $T : V \to V$ (translation operator).
- $|0\rangle \in V$ (vacuum).
- $Y(-,z) : V \to (\mathrm{End}\,V)[[z^{\pm 1}]]$ (state-field correspondence)

$Y(a,z) = \sum_{n\in\mathbb{Z}} z^{-n-1} a_{(n)}$

vertex operator of $a \in V$

OPE Operator Product Expansion.

$$[Y(a,z), Y(b,w)] = \sum_{n\geq 0} \left(\partial_w^n \delta(z,w)/n!\right) Y(a_{(n)}b, w)$$

or $Y(a,z)Y(b,w) \sim \sum_{n\geq 0} \dfrac{Y(a_{(n)}b, w)}{(z-w)^{n+1}}$ ← in physics literature

$$\Longleftrightarrow \quad [a_{(m)}, b_{(n)}] = \sum_{k\geq 0} (a_{(k)}b)_{(m+n-k)}.$$

Exm (affine vertex algebra, WZW model)

- $\mathfrak{g}$: finite dimensional simple Lie algebra / $\mathbb{C}$
- $\hat{\mathfrak{g}} = \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}K$: affine Lie algebra.

$$[a_m, b_n] = [a,b]_{m+n} + m\,\delta_{m+n,0}(a|b)\,K \qquad a_m := a \otimes t^m.$$

- For $k \in \mathbb{C}$,

$$V_k(\mathfrak{g}) := \mathrm{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g}[t]\oplus\mathbb{C}K} \mathbb{C}|k\rangle$$

where $a_m|k\rangle = 0 \ (a \in \mathfrak{g},\ m \geq 0)$, $K|k\rangle = k|k\rangle$

$V_k(\mathfrak{g})$ has a unique VA structure such that

- $|k\rangle$ is the vacuum. ← omitting the definition of T
- $Y(a_{-1}|k\rangle, z) = \sum_{n\in\mathbb{Z}} z^{-n-1} a_n$

Conformal vertex algebras

A vertex algebra which has the
"internal symmetry of the Virasoro algebra".

A CVA consists of

- V : vertex algebra
- $\omega \in V$, called Virasoro vector
- $c \in \mathbb{C}$, called central charge

such that

fundamental relation of Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3-m)\delta_{m+n,0}\, c$$

where $Y(\omega, z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n$.

Exm (Sugawara construction)

For $k \neq -\underset{\sim\sim}{h^\vee}$, dual Coxeter number of $\mathfrak{g}$.

$V_k(\mathfrak{g})$ is a CVA of central charge $c = \dfrac{k \cdot \dim \mathfrak{g}}{k + h^\vee}$

by the Virasoro vector

$$\omega = \frac{1}{2(k+h^\vee)} \sum_{a=1}^{\dim \mathfrak{g}} J_{a,-1} J^a_{-1} |k\rangle$$

where

- $\{J^a\}$: basis of $\mathfrak{g}$
- $\{J_a\}$: dual basis with respect to

$$(-|-) := \frac{1}{2h^\vee} (-|-)_{\text{Killing}} : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$$

$\llcorner$ normalised invariant form.

§. Geometric description of vertex algebras

- $(X, \mathcal{O}_X)$: smooth algebraic curve / $\mathbb{C}$
- V : (quasi) conformal vertex algebra.

One can construct the "VA bundle":

$$\mathcal{V}_X = \mathcal{A}ut_X \underset{\mathrm{Aut}\,\mathcal{O}}{\times} V \twoheadrightarrow X$$

c.f.) chapter 6 of [Frenkel, Ben-Zvi, 2001]

$$\mathcal{V}_{X,x} \cong V \quad \text{for each } x \in X.$$

Let $\mathcal{L}_X$ be the sheaf of sections of $\mathcal{V}_X$.

Then:

- For $x \in X$ and a local coordinate (U, z) of x,

$$\mathcal{L}_X|_U \cong V \underset{\mathbb{C}}{\otimes} \mathcal{O}_X|_U$$

The $\mathcal{O}_X$-module $\mathcal{L}_X$ has the flat connection ∇ defined as follows: flatness is trivial since X is a curve

For $x \in X$, take a local coordinate (U, z) of x, and define

$$\boxed{\mathcal{L}_X|_U \cong V \otimes_{\mathbb{C}} \mathcal{O}_X|_U}$$

$$\begin{aligned} \nabla_{\partial_z} : \mathcal{L}_X|_U &\longrightarrow \mathcal{L}_X|_U \\ v \otimes f &\longmapsto v \otimes \partial_z f + L_{-1} v \otimes f \end{aligned}$$

Virasoro operator of V

This gives rise to the connection ∇.

(Independent of the choice of local coordinates).

$\mathcal{L}_X$ has the left $\mathcal{D}_X$-module structure by ∇.

Let $\mathcal{L}_X^r$ be the corresponding right $\mathcal{D}_X$-module :

$$\mathcal{L}_X^r := \mathcal{L}_X \otimes_{\mathcal{O}_X} \omega_X .$$

$\mathcal{L}_X^r$ is not just a D-module.

It has the operation

$$\mu : (\mathcal{L}_X^r \boxtimes \mathcal{L}_X^r)(\infty\Delta) \longrightarrow \Delta_* \mathcal{L}_X^r$$

satisfying

$\Delta : X \to X^2$: diagonal embedding.

(1) (skew-symmetry) $\mu^{(2,1)} = -\mu$.

(2) (Jacobi identity) $\mu_{1,\{2,3\}} = \mu_{\{1,2\},3} + \mu_{2,\{1,3\}}$

c.f.) chapter 19 of [Frenkel-BenZvi].

Chiral algebra

Generalizing the structure of $\mathcal{L}_X^r$, we arrive at the notion of chiral algebras.

- $\mathcal{A}$: right $\mathcal{D}_X$-module.
- $\mu : (\mathcal{A} \boxtimes \mathcal{A})(\infty\Delta) \longrightarrow \Delta_* \mathcal{A}$, called chiral operation

satisfying "skew-symmetry" and "Jacobi identity".

To describe the composition of the chiral operation, Beilinson and Drinfeld introduced the chiral operad.

$$P^{ch}(n) = \mathrm{Hom}_{\mathcal{D}_{X^n}}\Big(\mathcal{A}^{\boxtimes n}\big(\infty \bigcup_{1 \le i < j \le n} \Delta_{i,j}^{(n)}\big),\ \Delta_*^{(n)} \mathcal{A}\Big).$$

$\Delta_{i,j}^{(n)} \subset X^n$: (i,j)-diagonal. $\Delta^{(n)} \subset X^n$: small diagonal

§. Operad

A device that describes the composition of operations.

Consider n-operations $f: V^{\otimes n} \to V$:

(1). Operations can be composited :

For $f_k: V^{\otimes m_k} \to V$ and $g: V^{\otimes n} \to V$, we have

$$g \circ (f_1 \otimes \cdots \otimes f_n) : V^{\otimes (m_1 + \cdots + m_n)} \longrightarrow V .$$

(2). There is an "unit" operation : $\mathrm{id}_V : V \to V$.

(3). The input of the operations can be swapped :

$$f: V^{\otimes 2} \to V \quad \rightsquigarrow \quad f^{(1,2)}: v \otimes w \longmapsto f(w \otimes v) .$$

In general, we have $f^{\sigma}: V^{\otimes n} \to V$

for $f: V^{\otimes n} \to V$ and $\sigma \in \mathfrak{S}_n$.

The notion of operads is a generalization of (1) ~ (3).

An **Operad** consists of

- $\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ where $\mathcal{P}(n)$: right $\mathbb{C}[\mathfrak{S}_n]$-module, called **$\mathfrak{S}$-module**

- For $n \in \mathbb{N}$ and $m_1, \cdots, m_n \in \mathbb{N}$, a linear map

$$\gamma_{m_1,\cdots,m_n} : \mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_n) \longrightarrow \mathcal{P}(m_1 + \cdots + m_n)$$
$$g \otimes x_1 \otimes \cdots \otimes x_n \longmapsto g \circ (x_1 \otimes \cdots \otimes x_n)$$

called **composition map**.

- An element $id \in \mathcal{P}(1)$, called **unit**.

Exm (endomorphism operad)

Let V be a linear space.

- For $n \in \mathbb{N}$,

$$\mathrm{End}_V(n) := \mathrm{Hom}_{\mathbb{C}}(V^{\otimes n}, V) \curvearrowleft \mathfrak{S}_n$$

$$f^{\sigma} : V^{\otimes n} \longrightarrow V, \quad v_1 \otimes \cdots \otimes v_n \longmapsto f(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)})$$

- For $n \in \mathbb{N}$ and $m_1, \cdots, m_n \in \mathbb{N}$,

$m := m_1 + \cdots + m_n$

$$\gamma_{m_1,\cdots,m_n} : \mathrm{End}_V(n) \otimes \mathrm{End}_V(m_1) \otimes \cdots \otimes \mathrm{End}_V(m_n) \longrightarrow \mathrm{End}(m)$$

$$g \otimes f_1 \otimes \cdots \otimes f_n \longmapsto g \circ \underbrace{(f_1 \otimes \cdots \otimes f_n)}_{: V^{\otimes m} \to V^{\otimes n}}$$

- The unit is $\mathrm{id}_V \in \mathrm{End}_V(1)$.

Algebraic operations have special symmetries.

For example, the Lie bracket $[-,-]$ of a Lie algebra $\mathfrak{g} = (V, [-,-])$ satisfies the skew-symmetry and the Jacobi identity.

To describe these symmetry we use the morphisms of operads.

<u>Exm</u> There are operads $\mathcal{C}om$, $\mathcal{A}ssoc$ and $\mathcal{L}ie$ such that

$$\mathrm{Hom}_{op}(\mathcal{C}om, \mathcal{E}nd_V) \cong \{\text{comm. alg. strs. on } V\}$$

$$\mathrm{Hom}_{op}(\mathcal{A}ssoc, \mathcal{E}nd_V) \cong \{\text{assoc. alg. strs. on } V\}.$$

$$\mathrm{Hom}_{op}(\mathcal{L}ie, \mathcal{E}nd_V) \cong \{\underbrace{\text{Lie alg. strs}}_{\text{i.e. Lie brackets}}\text{. on } V\}.$$

The operad of vertex algebras

- Beilinson and Drinfeld defined the structure of chiral algebra on a D_X-module A as a morphism $\alpha : \mathcal{L}ie \longrightarrow P^{ch}$.

$$P^{ch}(n) = \mathrm{Hom}_{D_{X^n}}\Big(A^{\boxtimes n}\big(\infty \bigcup_{1\le i<j\le n} \Delta_{i,j}^{(n)}\big), \Delta_*^{(n)} A\Big).$$

- Bakalov, De Sole, Heluani and Kac (2018) introduced an operad P^{ch}_{alg}, which is a purely algebraic translation of P^{ch}:

$$P^{ch}_{alg}(n) = \mathrm{Hom}_{D_n^T}\Big(V^{\otimes n} \otimes \underset{\sim\sim}{\mathcal{O}_n^{*T}}, V[\lambda_i]_{i=1}^n / \langle T + \lambda_1 + \cdots + \lambda_n \rangle\Big).$$

$$\mathcal{O}_n^{*T} = \mathbb{C}[(z_i - z_j)^{\pm 1}]_{1\le i<j\le n}$$

They proved:

$$\mathrm{Hom}_{op}(\mathcal{L}ie, P^{ch}_{alg}) \cong \{\text{VA str. on } (V, T)\}.$$

§. Concluding remarks.

- Conformal field theory with supersymmetry (SCFT) have also been actively studied in phisics.
- Heluani and Kac (2007) introduced the notion of **supersymmetric vertex algebras** (**SUSY VA**) to formulate the calculation of superfields which appeard in SCFT.
- Yanagida and I (2022) constructed an operad $P^{\mathrm{ch}N_W}$ (resp. $P^{\mathrm{ch}N_K}$), which encodes the structure of $N_W = N$ (resp. $N_K = N$) SUSY VA.