# String topology from the viewpoint of algebraic topology

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# Mapping spaces

$A$, $M$: topological space

**Definition**

$$\mathrm{Map}(A, M) := \{f\colon A \to M\} \quad \text{mapping space}$$

$\mathrm{Map}(A, M)$ is

- the space of all continuous maps from $A$ to $M$
- also a topological space (with compact-open topology)
- an $\infty$-dim space

**Problem in algebraic topology**

Study the homotopy type of $\mathrm{Map}(A, M)$

- homotopy type = "shape" of a space
- This is one of the fundamental problems in algebraic topology

# Free loop spaces

$M$: topological space

**Definition**

$$LM := \mathrm{Map}(S^1, M) \quad \text{the free loop space on } M$$

- $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$: the unit circle
- $LM$ is the space of all loops on $M$

**String topology**

Study algebraic structures on $H_*(LM)$
(under some assumptions on $M$)

- $H_*(LM)$: the homology of $LM$ (explained in the next page)

# Homology

$X$: topological space, $n \in \mathbb{N}$

## Homology

$H_n(X)$: the $n$-th homology of $X$

- $H_n(X)$ is a vector space which detects "$n$-dim holes" on $X$
- $\alpha \in H_n(X)$: $n$-dim "cycle" on $M$
- $\{H_n(X)\}_{n \in \mathbb{N}}$ reflects the "shape" of $X$

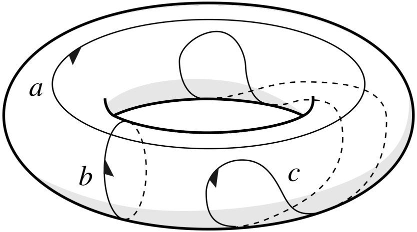


Figure: $H_1$(torus)

image from https://en.wikipedia.org/wiki/Homology_(mathematics)

# Loop product

$M$: oriented connected closed $m$-dim manifold
$LM = \mathrm{Map}(S^1, M)$: the free loop space on $M$

**Loop product [Chas-Sullivan '99]**

$$\mu\colon H_p(LM) \otimes H_q(LM) \to H_{p+q-m}(LM)$$

- $\alpha \in H_n(LM)$: $n$-dim cycle on $LM$
  i.e. $n$-dim ($n$-parmeter) family of loops on $M$
- Mixture of
  - $H_p(M) \otimes H_q(M) \to H_{p+q-m}(M)$: intersection product
  - $H_p(\Omega M) \otimes H_q(\Omega M) \to H_{p+q}(\Omega M)$: Pontrjagin product
    - induced by $\Omega M \times \Omega M \to \Omega M$ (loop concatenation)
    - $\Omega M := \mathrm{Map}_*(S^1, M)$: based loop space

# Construction of the loop product [Chas-Sullivan '99]

$M = \mathbb{R}^3$ (or 3-dim manifold)

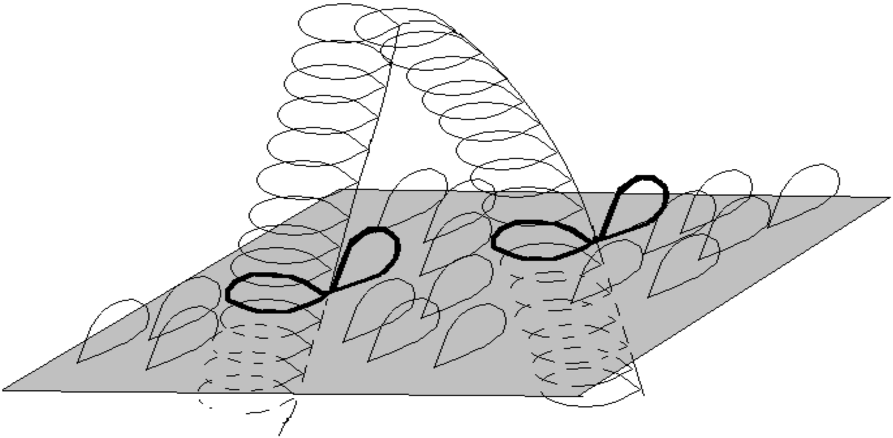

Figure: $H_1(LM) \otimes H_2(LM) \to H_{1+2-3}(LM)$

image from [Chas-Sullivan '99]

# Lift of the intersection product

$$\begin{array}{ccccc} LM \times LM & \xleftarrow{\mathrm{incl}} & LM \times_M LM & \xrightarrow{\mathrm{concat}} & LM \\ \downarrow & & \downarrow & & \\ M \times M & \xleftarrow{\quad\Delta\quad} & M & & \end{array}$$

**Definition (Loop product)**

$$\mu\colon H_*(LM \times LM) \xrightarrow{\mathrm{incl}^!} H_{*-m}(LM \times_M LM) \xrightarrow{\mathrm{concat}_*} H_{*-m}(LM)$$

$\Delta^!\colon H_*(M \times M) \to H_{*-m}(M)$: the intersection product

$\longrightarrow \mathrm{incl}^!\colon H_*(LM \times LM) \to H_{*-m}(LM \times_M LM)$: lift of $\Delta^!$

There are several ways to define $\mathrm{incl}^!$:

- Thom isomorphism,
- Eilenberg-Moore isomorphism ($H^*(LM \times_M LM) \cong \mathrm{Tor}(\cdots)$),
- etc. . .

# TQFT

**Theorem [Cohen-Godin '04]**

$H_*(LM)$ has the structure of 2-dim TQFT

TQFT = Topological Quantum Field Theory

- $\mu\colon H_*(LM) \otimes H_*(LM) \to H_*(LM)$: loop product
- $\delta\colon H_*(LM) \to H_*(LM) \otimes H_*(LM)$: loop coproduct
- $(\mathrm{id} \otimes \mu) \circ (\delta \otimes \mathrm{id}) = \delta \circ \mu = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta)$: Frobenius identity

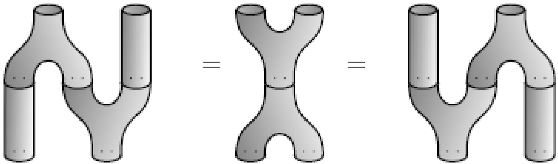

image from https://ncatlab.org/nlab/show/Frobenius+algebra

# Generalization

## String topology on Gorenstein spaces [Félix-Thomas '09]

Generalized loop (co)product to the case

$M$: Gorenstein space

- generalization of a manifold (from the viewpoint of Poincaré duality)
- e.g. $M = BG$: the classifying space of a Lie group $G$

## Brane topology

Generalize $LM$ to $L^k M := \mathrm{Map}(S^k, M)$

- $H_*(L^k M)^{\otimes 2} \to H_*(L^k M)$: brane product [Sullivan-Voronov '03]
- $H_*(L^k M) \to H_*(L^k M)^{\otimes 2}$: brane coproduct [W. '18]
  - The key to define the brane coproduct:
    $M \hookrightarrow L^{k-1}M$: embedding of "finite codimension" $(\dim \Omega^{k-1} M)$

# Thank you for your attention!