Quick introduction to chiral quantization

Shintaro Yanagida (Nagoya University)

2023/09/08

Shenzhen – Nagoya Workshop on Quantum Science 2023

1. Quantization in mathematical physics

Review talk on chiral quantization, partly based on

S.Y., "Derived gluing construction of chiral algebras",
Lett. Math. Phys., **111** (2021), article 51, 103pp.; arXiv:2004.10055.

1. Quantization in mathematical physics [3 pages]
   1.1. Quantization in general
   1.2. Deformation quantization
   1.3. Other notions of quantization
2. Vertex algebras and chiral quantization
3. Application: chiral quantization of Moore-Tachikawa TQFT

1.1. Quantization in general

- Let me use the word quantization to mean a mathematical formulation of the process of building quantum systems from classical mechanical/Hamiltonian systems.
- Canonical quantization (in physics).
  - For finite-dimensional mechanical system (first quantization):

    $$\{A, B\} \longmapsto \frac{1}{i\hbar}[\widehat{A}, \widehat{B}],$$

    replacing the Poisson bracket by commutators.
  - For field theory (second quantization), the procedure depends on the fields being quantized and the interaction.
- I first recall a well-known mathematical formulation of finite-dimensional case: deformation quantization.

1.2. Deformation quantization

For simplicity, I give an algebraic explanation.

- A classical Hamiltonian system can be encoded by a Poisson algebra $(A, \cdot, \{\cdot, \cdot\})$ consisting of c.f. Hayami-san's talk
  - $(A, \cdot)$: a (unital finitely-generated) commutative algebra with product $\cdot$ encoding the functions on the phase space of the classical system,
  - $\{\cdot, \cdot\}$: Poisson bracket, a bi-derivation (bilinear form with Leibniz rule) satisfying the Jacobi identity.

    $\{\cdot, \cdot\}$ is called symplectic if it is non-degenerate.
- Given a Poisson algebra $(A, \cdot, \{\cdot, \cdot\})$, a deformation quantization is a (non-commutative) algebra $(A[\![\hbar]\!] := \{\sum_{n=0}^{\infty} a_n \hbar^n \mid a_n \in A\}, \star)$ s.t.
  - $f \star g = f \cdot g + O(\hbar)$,
  - $[f, g] = \hbar\{f, g\} + O(\hbar^2)$, where $[f, g] := f \star g - g \star f$.

  A deformation quantization of a Poisson manifold is defined similarly.

  [F. Bayen, et. al., Ann. Phys., 1978].
- A universal formula of $\star$-product: Kontsevich's formula.

  [M. Kontsevich, LMP, 2003] c.f. Deligne conjecture in Prof. Kong's talk

1.3. Other notions of quantization

- Geometric quantization: another finite-dimensional quantization.
    - Prequantization: Given a symplectic manifold (= phase space), construct a line bundle L with connection.
    - Polarization: Construct a quantum Hilbert space H from L.
    - Half-form correction. c.f. Li-san's talk
- Feynman path integral: perturbative determination of field quantization (infinite-dimensional).
- There are other notions of quantization in mathematics.
    - Quantization of algebraic groups by Hopf algebras (quantum groups). c.f. Hattori-san's talk
    - Connes' noncommutative geometry involving C^*-algebras.
    - A version of quantization for functions is q-analogs.
- Chiral quantization is a combination of finite-dimensional and infinite-dimensional (field theory) cases.

2. Vertex algebras and chiral quantization

Recall the Kostant-Kirillov Poisson algebra $R^{KK}(\mathfrak{g}) = (R, \cdot, \{\cdot,\cdot\})$:

- $\mathfrak{g}$: a complex simple Lie algebra with Lie bracket $[\cdot,\cdot]$.
  $(R, \cdot) := \mathrm{Sym}(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}/S_n$: the symmetric algebra of $\mathfrak{g}$.
  $R \cong \mathbb{C}[\mathfrak{g}^*]$: the coordinate ring (function alg.) of the affine space $\mathfrak{g}^*$.
- $\{\cdot,\cdot\}: R \otimes R \to R$: Kostant-Kirillov Poisson bracket on R,
  uniquely determined by $\{x, y\} := [x, y]$ for $x, y \in \mathfrak{g}$,
  and $\{xa, b\} := \{x, b\}a + x\{a, b\}$ for $x \in \mathfrak{g}$ and $a, b \in R$.
- Example: $\mathfrak{g} = \mathfrak{sl}_2 = \mathbb{C}e + \mathbb{C}f + \mathbb{C}h$, $e = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $f = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, $h = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$.
  $R = \mathbb{C}[e, f, h]$, $\{e, f\} = h$, $\{h, e\} = e$, $\{h, f\} = -f$.

2.1. 1st example: ... and affine vertex algebra [2/2]

The chiral quantization of the Kostant-Kirillov Poisson algebra $R^{KK}(\mathfrak{g})$ is the affine vertex algebra $V_k(\mathfrak{g})$. c.f. Nishinaka-san's talk

- $\mathfrak{g}$: a complex simple Lie algebra.
  $\widehat{\mathfrak{g}} = \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}K$: the affine Lie algebra associated to $\mathfrak{g}$.
  (without grading operator D)
  $V_k(\mathfrak{g}) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} \mathbb{C}_k$
  with $\mathbb{C}_k$ the 1-dim. rep. where $\mathfrak{g}[t]$ acts trivially and K acts by k.
  ($k \in \mathbb{C}$: level, U: the universal enveloping algebra)
  It has a unique vertex algebra structure such that $\mathbf{1} := 1 \otimes 1$ is the vacuum vector and $Y(x_{(-1)}\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} x_{(n)} z^{-n-1}$, $x_{(n)} := x \otimes t^n$.
- There is a canonical Li filtration on the vertex algebra $V_k(\mathfrak{g})$ s.t. the C_2-Poisson algebra $R(V_k(\mathfrak{g}))$ coincides with the Kostant-Kirillov Poisson algebra $R^{KK}(\mathfrak{g})$. [Y. Zhu, JAMS, 1996]

2.2. Vertex algebras

c.f. Nishinaka-san's talks and

- A vertex algebra $(V, |0\rangle, T, Y)$ consists of
  - a linear space V, called state space,
  - an element $|0\rangle \in V$, called vacuum,
  - an endomorphism $T \in \operatorname{End} V$, called translation,
  - a linear map $Y(\cdot, z)\colon V \to (\operatorname{End} V)[\![z^{\pm 1}]\!]$ (state-field corresp.), denoted as $Y(a, z) = a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, for each $a \in V$,

  satisfying

  (i) $a(z)b \in V(\!(z)\!)$ for any $a, b \in V$, $\quad V(\!(z)\!) := \{\sum_{n=-k}^{\infty} v_n z^n \mid v_n \in V\}$,

  (ii) $Y(|0\rangle, z) = \mathrm{id}_V$, $a(z)|0\rangle = a + O(z)$ for any $a \in V$ (vacuum axiom),

  (iii) $T|0\rangle = 0$, $[T, a(z)] = \partial_z a(z)$ for any $a \in V$ (translation invariance),

  (iv) $\forall a, b \in V$, $\exists N_{a,b} \in \mathbb{Z}_{\geq 0}$ s.t. $(z-w)^{N_{a,b}}[a(z), b(w)] = 0$ (locality, $\iff$ operator product expansion in Nishinaka-san's talk).

- A vertex algebra can be regarded as a linear space V equipped with infinitely many binary operations $(a, b) \mapsto a_{(n)} b \quad (n \in \mathbb{Z})$.

- Li filtration of a vertex algebra $V = (V, |0\rangle, T, Y)$: [H. Li, CMP, 2005]

$$V = F^0V \supset F^1V \supset F^2V \supset \cdots,$$
$$F^pV := \left\langle (a_1)_{(-n_1)} \cdots (a_r)_{(-n_r)} v \mid a_i, v \in V,\ n_i \in \mathbb{Z}_{>0},\ \textstyle\sum_i n_i \geq p \right\rangle_{\mathrm{lin}}.$$

- The 0-th graded part

$$R(V) := F^0V/F^1V = V/C_2(V), \quad C_2(V) := \left\langle a_{(-2)}b \mid a, b \in V \right\rangle_{\mathrm{lin}}.$$

is a Poisson algebra, called Zhu's C_2-algebra. [Y. Zhu, JAMS, 1996]

$$\overline{a}\cdot\overline{b} := \overline{a_{(-1)}b}, \quad \{\overline{a}, \overline{b}\} := \overline{a_{(0)}b} \qquad (\overline{a} \in R(V) \text{ for } a \in V).$$

The Poisson scheme $\operatorname{Spec} R(V)$ is called the associated scheme.

Definition

A chiral quantization of a Poisson algebra R is a vertex algebra V such that $R(V)$ is isomorphic to R.

2.4. 2nd example: Slodowy slices and W-algebras [1/2]

$\mathfrak{g}$: complex simple Lie algebra.

- The affine vertex algebra $V_k(\mathfrak{g})$ is a chiral quantization of $R^{KK}(\mathfrak{g})$.
- The (regular) W-algebra $W_k(\mathfrak{g}, f_{\mathrm{reg}})$ is a chiral quantization of the Slodowy slice $S_{f_{\mathrm{reg}}}$. [T. Arakawa, IMRN, 2015]

Recollection of Slodowy slice and W-algebra:

- $f \in \mathfrak{g}$: a nilpotent element ($:\Leftrightarrow \mathrm{ad}(f) := [x, \cdot] \in \mathrm{End}(\mathfrak{g})$ is nilpotent).
  $\{e, f, h\} \subset \mathfrak{g}$: $\mathfrak{sl}_2$-triple, $\mathfrak{g}^e := \{x \in \mathfrak{g} \mid [x, e] = 0\}$: centralizer of e.
  $$S_f := f + \mathfrak{g}^e \subset \mathfrak{g} \simeq \mathfrak{g}^* \quad \text{via Killing form.}$$
  S_f with the Kostant-Kirillov Poisson structure is called the Slodowy slice.
- Example: $\mathfrak{g} = \mathfrak{sl}_2 = \{\left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right] \mid a, b, c \in \mathbb{C}\}$, $f = f_{\mathrm{reg}} := \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$,
  $e = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $\mathfrak{g}^e = \mathbb{C}e$.
  $$S_{f_{\mathrm{reg}}} = f_{\mathrm{reg}} + \mathfrak{g}^e = \left[\begin{smallmatrix} 0 & * \\ 1 & 0 \end{smallmatrix}\right].$$
- Given a nilpotent element $f \in \mathfrak{g}$ and level $k \in \mathbb{C}$, we can construct a vertex algebra $W_k(\mathfrak{g}, f)$ called the W-algebra.
- Example: $\mathfrak{g} = \mathfrak{sl}_2$, $f = f_{\mathrm{reg}}$, $W_k(\mathfrak{sl}_2, f_{\mathrm{reg}})$ = the Virasoro vertex algebra.
  $[L_m, L_n] = L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$.

2.4. 2nd example: Slodowy slices and W-algebras

- $V_k(\mathfrak{g})$ is a chiral quantization of $R^{KK}(\mathfrak{g})$.
- $W_k(\mathfrak{g}, f_{\mathrm{reg}})$ is a chiral quantization of $S_{f_{\mathrm{reg}}}$.

These two chiral quantizations are related under Hamiltonian reduction.

$$
\begin{array}{ccc}
V_k(\mathfrak{g}) & \xrightarrow[\text{taking } C_2\text{-algebra}]{R(\cdot)} & R^{KK}(\mathfrak{g}) \\
\text{quantum Hamiltonian reduction} \big\downarrow & \circlearrowright & \big\downarrow \text{classical Hamiltonian reduction} \\
W_k(\mathfrak{g}, f_{\mathrm{reg}}) & \xrightarrow[\text{taking } C_2\text{-algebra}]{R(\cdot)} & S_{f_{\mathrm{reg}}}
\end{array}
$$

c.f. quantum Hamiltonian reduction = BRST reduction in Hayami-san's talk

2.5. Existence theorem of chiral quantization

Theorem

For any Poisson algebra R, there exists a vertex algebra V such that $R(V) \cong R$, i.e., a chiral quantization of R exists.

- For any R, the arc algebra $R[\![t]\!] = \{\sum_{n=0}^{\infty} a_n t^n \mid a_n \in R\}$ has the structure of level 0 Poisson vertex algebra.

  [T. Arakawa, Math. Z., 2012]

- For any Poisson vertex algebra P, there exists a vertex algebra V such that $\mathrm{gr}\, V = P$. c.f. [Tamarkin, PICM, 2002], chiral Deligne conjecture

- The associated graded space $\mathrm{gr}\, V := \bigoplus_{n=0}^{\infty} F^n V / F^{n+1} V$ of Li filtration of any vertex algebra V has a structure of Poisson vertex algebra.

  c.f. Hayami-san's talk

Open problem

$\exists$? explicit description of the above chiral quantization
(like Kontsevich's formula of deformation quantization)

3. Application: chiral quantization of Moore-Tachikawa TQFT

3.1. Moore-Tachikawa 2d Topological QFT

[G. Moore, Y. Tachikawa, String-Math 2011; arXiv:1106.5698]

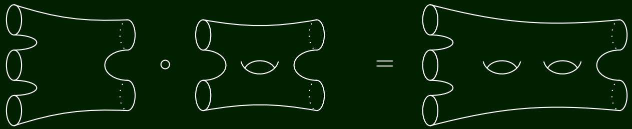
Moore and Tachikawa conjectured the existence of a functor

$$\eta_G : \mathrm{Bo}_2 \longrightarrow \mathrm{HS}$$

between certain symmetric monoidal categories with duality.

The source category Bo_2 is the 2-bordism category.

- Objects: $(S^1)^n$ for $n \in \mathbb{Z}_{\geq 0}$, identified with n.
- Morphisms: $\Sigma_{g,n_1+n_2} : n_1 \to n_2$, 2-dim. oriented manifolds with genus g and boundary $(S^1)^{n_1} \sqcup -(S^1)^{n_2}$.
- Composition := gluing.



$(\Sigma_{0,2+3} : 2 \to 3) \circ (\Sigma_{1,2+2} : 2 \to 2) = (\Sigma_{2,2+3} : 2 \to 3)$

- $\otimes := \sqcup$, disjoint union of manifolds.

3.1. Moore-Tachikawa 2d Topological QFT

The target HS is the category "of holomorphic symplectic varieties":

- Objects: semisimple algebraic groups over $\mathbb{C}$.
- Morphisms: $X\colon G_1 \to G_2$, holomorphic symplectic variety X with Hamiltonian $G_1 \times G_2$-action.

  $G \curvearrowright (Y, \omega)$ is Hamiltonian if $\exists \mu\colon Y \to \mathfrak{g}^* := \mathrm{Lie}(G)^*$, the moment map, s.t. $\langle d\mu(\cdot), \xi\rangle = -\iota_{\xi_Y}\omega$ with $\xi_Y(y) := \frac{d}{dt} e^{t\xi}.y\Big|_{t=0}$ for $\xi \in \mathfrak{g}$, and $\mu(g.y) = \mathrm{ad}^*_{g^{-1}}\,\mu(y)$ for $g \in G$.

  The identity morphism $\mathrm{id}_G := T^*G = G \times \mathfrak{g}^*$.
- Composition: For $X_{12} \in \mathrm{Hom}_{\mathrm{HS}}(G_1, G_2)$ and $X_{23} \in \mathrm{Hom}_{\mathrm{HS}}(G_2, G_3)$,

  $$X_{23} \circ X_{12} := (X_{12}^{\mathrm{op}} \times X_{23}) /\!\!/_{\mu} \Delta(G_2) = \mu^{-1}(0)/\Delta(G_2).$$

  $/\!\!/_{\mu}$: Hamiltonian reduction (symplectic quotient) for the moment map

  $$\mu\colon X_{12} \times X_{23} \to \mathfrak{g}_2^* := \mathrm{Lie}(G_2)^*, \quad \mu(x, y) := -\mu_{12}(x) + \mu_{23}(y)$$

  with μ_{12} the $\mathfrak{g}_2^*$-component of momentum map $X_{12} \to \mathfrak{g}_1^* \times \mathfrak{g}_2^*$.
- $\otimes$: given by Cartesian product.

3.1. Moore-Tachikawa 2d Topological QFT [3/3]

Moore and Tachikawa conjectured that, for each 1-conn. semisimple G, there exists a functor $\eta_G \colon \mathrm{Bo}_2 \to \mathrm{HS}$ with $\eta_G(n) = G^n$ and

$$\eta_G(\Sigma_{g,n_1+n_2}) : \text{holo. symplectic variety with Ham. } G^{n_1+n_2}\text{-action}$$
$$\text{(Moore-Tachikawa symplectic variety).}$$

A functor from Bo_2 is called a 2d topological QFT (Atiyah-Segal), and η_G is called Moore-Tachikawa TQFT. c.f. Wakatsuki-san's talk

The functoriality of η_G means that taking symplectic quotients of $\eta_G(\Sigma)$'s is compatible with gluing bordisms Σ's.

$$\eta_G(\Sigma'_{g',n_2+n_3} \circ \Sigma_{g,n_1+n_2}) \overset{\text{gluing}}{=\!=\!=} \eta_G(\Sigma''_{g'',n_1+n_3})$$

$$\Big\Vert \ \text{functoriality}$$

$$\eta_G(\Sigma'_{g',n_2+n_3}) \circ \eta_G(\Sigma_{g,n_1+n_2}) = \big(\eta_G(\Sigma_{g,n_1+n_2})^{\mathrm{op}} \times \eta_G(\Sigma'_{g',n_2+n_3})\big) /\!\!/ \Delta(G^{n_2})$$

3.2. BFN construction of η_G [1/2]

[A. Braverman, M. Finkelberg, H. Nakajima, Adv. Theor. Math. Phys., 2019]

Theorem (Braverman-Finkelberg-Nakajima)

Moore-Tachikawa 2d TQFT η_G exists.

- They introduced, in some equivariant derived constructible category $D_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ on the affine Grassmannian

$$\mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}, \quad G_{\mathcal{O}} := G\big(\mathbb{C}[\![z]\!]\big),\ G_{\mathcal{K}} := G\big(\mathbb{C}(\!(z)\!)\big),$$

  two distinguished objects $\mathcal{A}, \mathcal{B} \in D_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ which are ring objects with respect to the convolution product $\star$.

- Using these ring objects for the Langlands dual G^L, they showed that

$$\eta_G(\Sigma_{g,n}) := \mathrm{Spec}\Big(H^*_{G^L_{\mathcal{O}}}(\mathrm{Gr}_{G^L}, i^!_{\Delta}(\mathcal{A}^{\boxtimes n} \boxtimes \mathcal{B}^{\boxtimes g})), \star\Big)$$

  has a symplectic structure, and satisfies the gluing condition $\eta_G(\Sigma \circ \Sigma') \simeq \eta_G(\Sigma) \circ \eta_G(\Sigma')$.

3.2. BFN construction of η_G [2/2]

A few varieties in genus zero part can be described explicitly.

Denoting $W_G^n := \eta_G(\Sigma_{g=0,n})$, the gluing condition gives

$$W_G^n \circ W_G^m \simeq W_G^{n+m-2}.$$

- The case $n = 2$ is already explained:

$$W_G^2 = \eta_G\left(\right)$$  $$= \mathrm{id}_G = T^*G = G \times \mathfrak{g}^*.$$

- The case $n = 1$ is a bit non-trivial.

$$W_G^1 = \eta_G\left(\right)$$  $$= \eta_G\left(\right)$$  $$= G \times S_{f_{\mathrm{reg}}}$$

  with $S_{f_{\mathrm{reg}}} \subset \mathfrak{g}^*$ the Slodowy slice of the regular nilpotent $f_{\mathrm{reg}} \in \mathfrak{g}$.

- The case $n = 3$ for $G = \mathrm{SL}_2$ and SL_3 is

$$W_{\mathrm{SL}_2}^3 = (\mathbb{C}^2)^{\times 3}, \quad W_{\mathrm{SL}_3}^3 = \overline{O_{\mathrm{min}}} \text{ in } E_6.$$

$\overline{O_{\mathrm{min}}}$: closure of coadjoint orbit of minimal nilpotent element

3.3. Arakawa's chiral quantization $\eta^{\mathrm{ch}}_{G,g=0}$ [1/2]

[T. Arakawa, arXiv:1811.01577]

- Arakawa considered "chiral quantization" of η_G:

$$\eta^{\mathrm{ch}}_G : \mathsf{Bo}_2 \longrightarrow \mathsf{HS}^{\mathrm{ch}}.$$

  - Target category $\mathsf{HS}^{\mathrm{ch}}$:
    - Objects: semisimple algebraic groups (the same as HS).
    - Morphisms $V\colon G_1 \to G_2$: vertex algebras V equipped with

      $V_{-h_1^\vee}(\mathfrak{g}_1) \otimes V_{-h_2^\vee}(\mathfrak{g}_2) \to V$ (+ some cond.).
    - Composition of $V_{12}\colon G_1 \to G_2$ and $V_{23}\colon G_2 \to G_3$:

$$V_{23} \circ V_{12} := H^{\frac{\infty}{2}+0}(\widehat{\mathfrak{g}}_{-2h_2^\vee}, \mathfrak{g}_2, V_{12}^{\mathrm{op}} \otimes V_{23}),$$

$H^{\frac{\infty}{2}+*}(\cdot,\cdot,\cdot)$: relative BRST (semi-infinite) cohomology

(quantum Hamiltonian reduction)

  - The functor η^{ch}_G should sit in a commutative diagram

$$\begin{array}{ccc} \mathsf{Bo}_2 & \overset{\eta^{\mathrm{ch}}_G}{\dashrightarrow} & \mathsf{HS}^{\mathrm{ch}} \\ \Big\| & & \Big\downarrow {\scriptstyle \mathrm{Spec}\, R(-) \text{ taking associated scheme}} \\ \mathsf{Bo}_2 & \xrightarrow{\eta_G} & \mathsf{HS} \end{array}$$

3.3. Arakawa's chiral quantization $\eta^{\mathrm{ch}}_{G,g=0}$ [2/2]

- Arakawa built genus 0 part $\eta^{\mathrm{ch}}_{G,g=0}\colon \mathsf{Bo}_2|_{g=0} \to \mathsf{HS}^{\mathrm{ch}}$.

Theorem (Arakawa)

$\exists$ a family $\{V^{\mathcal{S}}_{G,n} = \eta^{\mathrm{ch}}_{G,g=0}(\Sigma_{g=0,n}) \mid n \in \mathbb{Z}_{\geq 0}\}$ of vertex algebras s.t.

$$V^{\mathcal{S}}_{G,1} \simeq H^0_{\mathrm{DS}}(\mathcal{D}^{\mathrm{ch}}_G), \quad V^{\mathcal{S}}_{G,2} \simeq \mathcal{D}^{\mathrm{ch}}_G, \quad V^{\mathcal{S}}_{G,m} \circ V^{\mathcal{S}}_{G,n} \simeq V^{\mathcal{S}}_{G,m+n-2},$$

and their associated schemes are Moore-Tachikawa symplectic varieties:

$$W^n_G \simeq \mathrm{Spec}\, R(V^{\mathcal{S}}_{G,n}).$$

- As a corollary, Beem-Rastelli conjecture [C. Beem, L. Rastelli, JHEP, 2018]

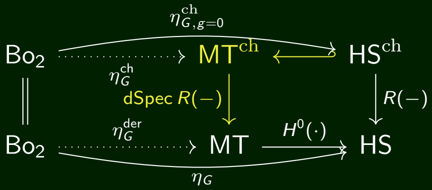
$$\mathcal{M}_{\mathrm{Higgs}}(\mathcal{T}) \overset{?}{\simeq} \mathrm{Specm}\, R(V(\mathcal{T})) \quad \forall\, \mathcal{T}\colon \mathcal{N}=2 \text{ 4d SCFT}$$

$$V\colon \{\text{4d } \mathcal{N}=2 \text{ SCFTs}\} \longrightarrow \{\text{conformal vertex algebras}\}$$

is affirmatively solved for genus 0 class $\mathcal{S}$ theories $\mathcal{T} = \mathcal{T}^{\mathcal{S}}_{\Sigma_{0,n}}$.

3.4. Toward higher-genus quantization

- In order to extend Arakawa's functor $\eta^{\mathrm{ch}}_{G,g=0}$ to the case $g > 0$, the target category $\mathrm{HS}^{\mathrm{ch}}$ should be enlarged.
  I built such an enlarged target. [S.Y., Lett. Math. Phys., 2021].
- I constructed an ∞-category $\mathrm{MT}^{\mathrm{ch}}$ which will be the target of the extension η^{ch}_{G} of Arakawa's $\eta^{\mathrm{ch}}_{G,g=0}$. This $\mathrm{MT}^{\mathrm{ch}}$ sits in the following commutative diagram.



- $\mathrm{MT}^{\mathrm{ch}}$ is designed to give a "chiral quantization" of the ∞-category MT of derived Moore-Tachikawa varieties. [D. Calaque, 2015]

- The ∞-category MT of derived Moore-Tachikawa varieties [Calaque]:
  - Objects: semisimple algebraic groups (same as HS)
  - Morphisms $X\colon G_1 \to G_2$: derived Poisson scheme X with Hamiltonian $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$-action.

    c.f. Hayami-san's talk
  - Composition of $X_{12} \in \mathrm{Map}_{\mathrm{MT}}(G_1, G_2)$ and $X_{23} \in \mathrm{Map}_{\mathrm{MT}}(G_2, G_3)$:

$$X_{23} \,\widetilde{\circ}\, X_{12} := \left(X_{12}^{\mathrm{op}} \otimes X_{23}\right) /\!\!/_{\mu}^{\mathbb{L}} \mathrm{Sym}(\mathfrak{g}_2).$$

    $/\!\!/_{\mu}^{\mathbb{L}}$: derived Hamiltonian reduction of derived Poisson schemes $\mu := -\mu_{12}^2 \otimes 1 + 1 \otimes \mu_{23}^1$. The composition $\widetilde{\circ}$ is called derived gluing.
- The ∞-category $\mathrm{MT}^{\mathrm{ch}}$ [Y.]:
  - Objects: semisimple algebraic groups (same as $\mathrm{HS}, \mathrm{HS}^{\mathrm{ch}}$).
  - 1-Morphisms: dg vertex algebras V with $\mu_V\colon V_k(\mathfrak{g}_1) \otimes V_l(\mathfrak{g}_2) \to V$.
  - Compositions of $V_{12}\colon G_1 \to G_2$ and $V_{23}\colon G_2 \to G_3$ is given by derived quantum Hamiltonian reduction:

$$V_{23} \,\widetilde{\circ}\, V_{12} := \mathrm{BRST}(\widehat{\mathfrak{g}}_{l+m}, V_{12}^{\mathrm{op}} \otimes V_{23}, \mu) \qquad \text{(chiral derived gluing)}.$$

3.4. Toward higher-genus quantization

Theorem ([S.Y., LMP, 2021])

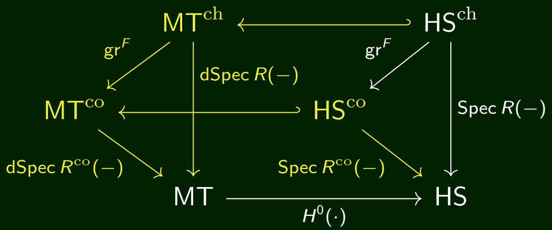
Taking derived associated scheme gives a functor

$$\mathrm{dSpec}\, R(-)\colon \mathsf{MT}^{\mathrm{ch}} \longrightarrow \mathsf{MT},$$

i.e., $\mathrm{dSpec}\, R(V \mathbin{\widetilde{\circ}} W) \simeq \mathrm{dSpec}\, R(V) \mathbin{\widetilde{\circ}} \mathrm{dSpec}\, R(W)$.

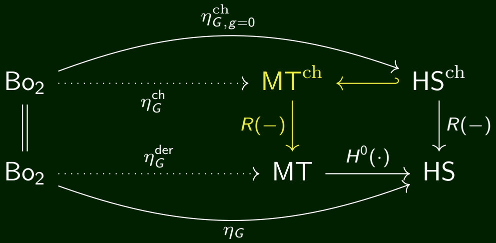
I also constructed an ∞-category $\mathsf{MT}^{\mathrm{co}}$ of dg Poisson vertex algebras and related functors, which sit in the following commutative diagram:

c.f. Hayami-san's talk



3.4. Toward higher-genus quantization

- I expect the existence of the functors η_G^{ch} and η_G^{der} making the following diagram commute:



Open problem

Describe dg vertex algebras $\eta_G^{\mathrm{ch}}(\Sigma_{g>0,n}) \in \mathrm{MT}^{\mathrm{ch}}$,
in particular $\eta_G^{\mathrm{ch}}(\Sigma_{1,1})$, explicitly.

Thank you.