Introduction to the q-deformed Virasoro algebra

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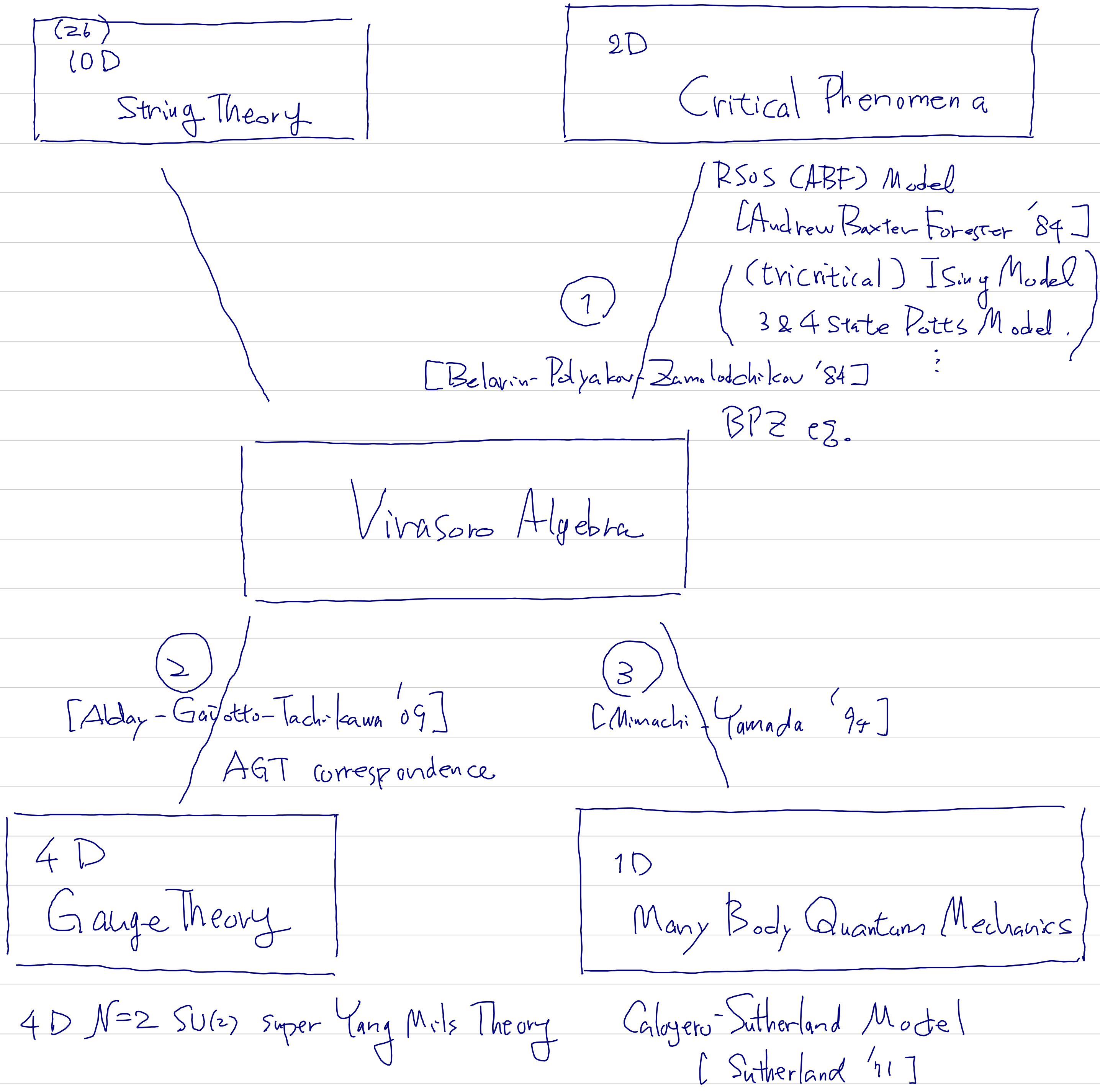
0 Introduction



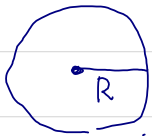
{2?}

11D

M-Theory

[Witten '95]

11-dim, S^1



$z = e^R$

2D

non-Critical Phenomena

RSOS (ABF) Model

[Andrew Baxter Forester '84]

(1)

[Lukyanov-Pugai '96]

[Shakirov '21]

z-BPZ eq.

z- Virasoro Algebra

[Kubo-Odake-Shiraishi-H.A. '95]

(2)

[Yamada-H.A. '09]

(3)

[Kubo-Odake-Shiraishi-H.A. '95]

5D

Gauge Theory

5D $\mathcal{N}=2$ SU(2) super Yang Mils Theory

1D Relativistic

Many Body Quantum Mechanics

Ruijenaars Model

[Ruijenaars '87]

1 Virasoro Algebra (Conformal Field Theory CFT)

* Virasoro operator $\{L_n\}_{n \in \mathbb{Z}}$

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} n (n^2-1) \delta_{n+m,0}$$

$[A,B] := AB - BA$

$\delta_{n,m} = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$

$c \in \mathbb{C}$ central charge

Rem. $\langle i L_{-1}, -L_0, i L_1 \rangle \simeq SU(2)$: angular Momentum

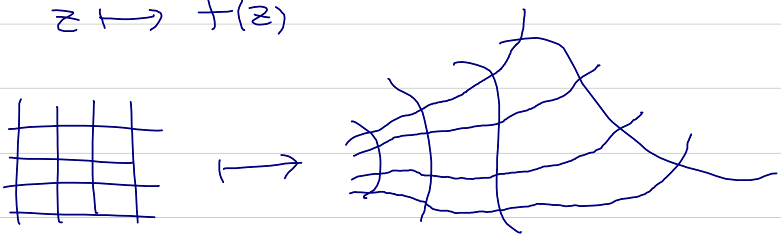
$i := \sqrt{-1}$

$$\begin{cases} [L_0, L_{\pm 1}] = \mp L_{\pm 1} \\ [L_1, L_{-1}] = 2 L_0 \end{cases}$$

Rem. $c = 0 \Rightarrow$ Conformal transformation (local Scale transformation)

$L_n \simeq - z^{n+1} \frac{\partial}{\partial z}$ (classical symmetry of the String theory)

$\mathbb{C} \ni z \longmapsto f(z)$



Rem. $\frac{\partial}{\partial z} z^n = n z^{n-1} + z^n \frac{\partial}{\partial z}$

$D z^n = n z^n \quad z^n D \qquad D := z \frac{\partial}{\partial z}$

$= z^n (n + D)$

$z^n D z^m D = z^{n+m} (m + D) D$

$-) \; z^m D z^n D = z^{n+m} (n + D) D$

$[z^n D, z^m D] = z^{n+m} (m-n) D = (m-n) z^{n+m} D$

2 Primary Operators $\{\Phi_h(z)\}$ $h, z \in \mathbb{C}$

$$[L_n, \Phi_h(z)] = z^n \left\{ z\frac{\partial}{\partial z} + (n+1)h \right\} \Phi_h(z), \quad h \in \mathbb{C} : \text{Scale dim.}$$

Rem. ($-L_n$ of $c=0$)

$h=0 \Rightarrow$
$$[L_n,[L_m,\Phi]] = [L_n, z^m D\Phi] = z^m D z^n D \Phi$$
$$-)\ [L_m,[L_n,\Phi]] = [L_m, z^n D\Phi] = z^n D z^m D \Phi$$
$$[[L_n,L_m],\Phi] = [z^m D, z^n D]\Phi = (n-m) z^{n+m} D \Phi$$

Rem. Operators in 2D Critical Phenomena

Critical Phenomena	c	h
Ising	1/2	1/16 1/2
tricritical Ising	7/10	1/10 3/5 3/2 3/80 7/16
3-States Potts	4/5	...
⋮		

* Vacuum $|h\rangle$ $\begin{cases} L_n|h\rangle = 0, \\ L_0|h\rangle = h|h\rangle \end{cases}$ $\begin{cases} \langle 0|L_{-n} = 0 \\ \langle h|L_0 = h\langle h| \end{cases}$ $(h\in\mathbb{C})$ $(n>0)$

* Fact I.

Correlation function $\langle h_\infty | \Phi_{h_2}(1)\, \Phi_{h_1}(z) | h_0\rangle$ ← z by $sl(2,\mathbb{C})$ sym.

⇓

easily computed ①

by $\begin{cases} 1) \text{ differential eq. (BPZ eq.) } 2^{nd} \text{ order diff. eq. w/ regular singular points (Fuchsian, hypergeometric)} \\ 2) \text{ Free Boson realization} \end{cases}$

Rem. $\langle h_\infty|\Phi_h(z)|h_0\rangle = z^{h_\infty - h - h_0} f(h_\infty, h, h_0)$: almost trivial

$\left(\because \langle h_\infty|[L_0,\Phi_h(z)]|h_0\rangle = (z\frac{\partial}{\partial z} + h)\langle h_\infty|\Phi_h(z)|h_0\rangle = (h_\infty - h_0)\langle h_\infty|\Phi_h(z)|h_0\rangle \right)$

* Fact II.

$\langle h_\infty|\Phi_{h_2}(1)\,\Phi_{h_1}(z)|h_0\rangle$ = Partition Function of 4D $\mathcal{N}=2$ SU(2) SYM

AGT - correspondence ②

[3] Virasoro States

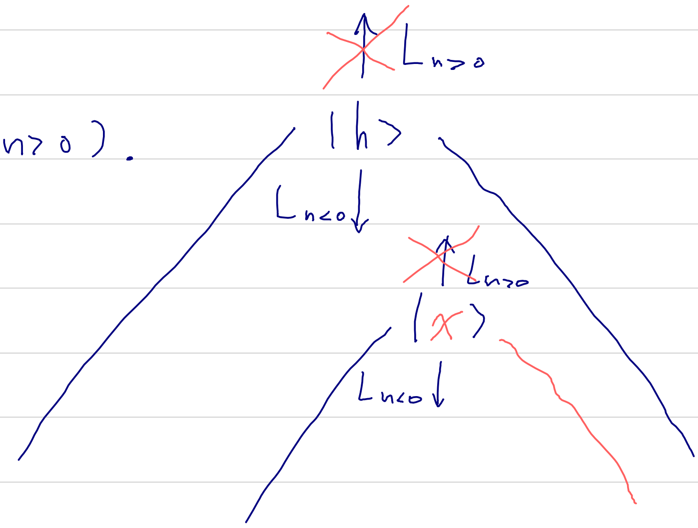
$$\left(\begin{cases} L_n |h\rangle = 0 & (n>0) \\ L_0 |h\rangle = h|h\rangle \end{cases} \right)$$

$$|h\rangle$$
$$L_{-1}|h\rangle$$
$$L_{-2}|h\rangle,\ L_{-1}L_{-1}|h\rangle$$
$$L_{-3}|h\rangle,\ L_{-2}L_{-1}|h\rangle,\ L_{-1}L_{-1}L_{-1}|h\rangle$$
$$L_{-4}|h\rangle,\ L_{-3}L_{-1}|h\rangle,\ L_{-2}L_{-2}|h\rangle,\ L_{-2}L_{-1}L_{-1}|h\rangle,\ L_{-1}L_{-1}L_{-1}L_{-1}|h\rangle$$
$$\vdots$$

✻ For a special c & h, ⇓

$\exists$ Singular state $|\chi\rangle$,

s.t. $L_n |\chi\rangle = 0 \quad (n>0)$.



✻ Fact III.

{Singular states} $\overset{\exists}{\hookrightarrow}$ {Excited States of Calogero-Sutherland Model}

[Mimachi-Yamada '94] (3)

Rem. $W_3, W_4 \cdots$ algebras case ($sl_3, sl_4, \cdots$ type extensions of Virasoro)

$\{ \quad \vdots \quad \} \overset{\cong}{\longleftrightarrow} \{ \quad \vdots \quad \}$

[Matsuo, Odake-Shiraishi-H.A '94]

[4] q-deformed Virasoro algebra (q-deformed CFT)

* q-Virasoro operator $\{T_n\}_{n\in\mathbb{Z}}$

$$[T_n, T_m] = -\sum_{\ell\geq 0} f_\ell \left(T_{n-\ell}T_{m+\ell} - T_{m-\ell}T_{n+\ell} \right) - \frac{(1-q)(1-t^{-1})}{1-p}(p^n - p^{-n})\,\delta_{n+m,0} \quad (*)$$

$q, t, p \in \mathbb{C}$

$$\begin{cases} t = q^\beta \\ p = q/t \end{cases} \qquad \beta \in \mathbb{C} \sim \text{central charge}$$

$$f(x) := \sum_{\ell\geq 0} f_\ell x^\ell := \exp\left\{ \sum_{n>0} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})}{1+p^n} x^n \right\}$$

Rem. $(q := e^{\hbar} \to 1)$ $(\hbar \to 0)$ $\Longrightarrow$ Virasoro w/ $c = 1-6(\beta-1)(1-\frac{1}{\beta})$

$$\begin{cases} T(z) := \sum_{n\in\mathbb{Z}} T_n z^{-n} \quad \text{generating function.} \\ \quad = (p^{\frac{1}{2}} + p^{-\frac{1}{2}}) + \beta z^2 L(z)\hbar^2 + \mathcal{O}(\hbar^4) \\ L(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \end{cases} \qquad \begin{cases} f_0 = 1 \\ f_{n>0} = \mathcal{O}(\hbar^2) \end{cases}$$

Rem. By generating func.

$$(*) \Longleftrightarrow f\left(\frac{w}{z}\right) T(z) T(w) - T(w) T(z) f\left(\frac{z}{w}\right) = -\frac{(1-q)(1-t^{-1})}{1-p}\left(\delta\left(\frac{wp}{z}\right) - \delta\left(\frac{w}{zp}\right) \right) \quad (**)$$

$$\begin{cases} \delta(x) := \sum_{n\in\mathbb{Z}} x^n \\ f(x) f(x) = \delta(x) f(1) \end{cases}$$

[5] Free Boson realization.

* Free Boson operator $\{a_n, Q\}_{n\in\mathbb{Z}}$ $(p := q/t)$

$$[a_n, a_m] = -\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+p^n}\delta_{n+m,0}$$
$$[a_n, Q] = \frac{1}{2}\delta_{n,0}$$

* q-Virasoro operator

$$\begin{cases} a_n|\tilde{h}\rangle = 0 \quad (n>0) \\ e^{-2\sqrt{\beta}a_0}|\tilde{h}\rangle = h|\tilde{h}\rangle \end{cases}$$

$$\begin{cases} T(z) = \Lambda_+(z) + \Lambda_-(z) \\ \Lambda_\pm(z) := :e^{\pm\sum_{n\neq 0} a_n p^{\pm\frac{n}{2}} z^{-n}}: \, q^{\pm\sqrt{\beta}a_0} p^{\pm\frac{1}{2}} \end{cases} \Longrightarrow (**)$$

$:*:$: Normal ordering (+ mode moves to the right)

e.g. $:a_3 a_{-3}: = a_{-3}a_3$

$:a_0 Q: = Q a_0$

* Primary operator $\tilde{\Phi}_h(z)$ [Yamada-H.A '10] $t^\gamma := h$

$$\tilde{\Phi}_h(z) := :\exp\left\{\sum_{n\neq 0}\frac{(h^{\frac{n}{2}} - h^{-\frac{n}{2}})a_n z^{-n}}{(q^{\frac{n}{2}}-q^{-\frac{n}{2}})(t^{\frac{n}{2}}-t^{-\frac{n}{2}})}\right\}: e^{-\gamma\sqrt{\beta}Q} z^{-\gamma\sqrt{\beta}a_0}$$

$$\Longrightarrow \begin{cases} g_h^L\left(\frac{w}{z}\right)\Lambda_\pm(z)\tilde{\Phi}_h(w) - \tilde{\Phi}_h(w)\Lambda_\pm(z)g_h^R\left(\frac{z}{w}\right) = (h^{-1})\,\delta\left(\frac{w}{z h^{\pm}}\right):\Lambda_\pm(z)\tilde{\Phi}_h(w): \\ g_h^L(x) := \cdots \end{cases}$$

Rem. $h = t, q^{-1} \Longrightarrow \tilde{\Phi}_t(z), \tilde{\Phi}_{q^{-1}}(z)$: degenerate operator

[6] q-deformation of (1)(2)(3)

☆ Fact I′:

Correlation function of $\tilde{\Phi}_h(z)$'s (w/ Screening Current) (1′)

= Correlation function of ABF model [Lukyanov-Pugai '96]

☆ Fact II′:

$\langle \tilde{h}_\infty | \tilde{\Phi}_{h_2}(1) \tilde{\Phi}_{h_1}(z) | \tilde{h}_0 \rangle$ = Partition function of 5D $\mathcal{N}=2$ SU(2) SYM (2′)

[Yamada-H.A '09] [Feigin-Shiraishi-H.A '11] [Fukuda-Ohkubo-Shiraishi '19]

☆ Fact III′:

$\{\text{Singular states}\} \overset{\exists}{\hookrightarrow} \{$ Excited States of Ruijsenaars Model $\}$

[Kubo-Odake-Shiraishi-H.A '95] (3′)

Rem. q-W_3, q-W_4, $\cdots \Rightarrow \cong$ [Odake-Shiraishi-H.A '96]

☆ Fact I″: q-BPZ eq [Shakirov '21] (1″)

$$\Psi(z_1, z_2) := \langle \tilde{h}_\infty | \tilde{\Phi}_{h_3}(1) \tilde{\Phi}_{+}(z_2) \tilde{\Phi}_{h_1}(z_1) | \tilde{h}_0 \rangle$$

($\tilde{\Phi}_{+}$ ← degenerate operator)

$\Downarrow$

$$\Psi(z_1,z_2) = \frac{1}{\varphi(z_1)\varphi(z_2)}\, \mathcal{B}\, \frac{\varphi(\bullet)\varphi(\bullet)}{\varphi(\bullet z_1)\varphi(\bullet z_1)\varphi(\bullet z_2)\varphi(\bullet z_2)}\, \mathcal{B}\, \frac{1}{\varphi(\bullet z_1)\varphi(\bullet z_2)}\, \Psi(\bullet z_1, \bullet z_2)$$

(shift)

$$\begin{cases} \varphi(x) := \prod_{k \geq 0} (1 - q^k x) \\ \mathcal{B} := q^{\frac{1}{2}(D_2 - D_1)^2} \quad : \ q\text{-Borel transformation op} \\ D_i := z_i \frac{\partial}{\partial z_i} \end{cases}$$

Proof. [Hasegawa-Kanno-Ohkawa-Shakirov-Shiraishi-Yamada-H.A '23]

By reducing to q-KZ eq. for $U_q(\widehat{sl}_2)$.