SUSTech-Nagoya workshop on Quantum Science

Pontryagin Duality For 2-Groups

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Pontryagin duality is an important notion to study representation theory of groups.

For any Abelian group A, one can define its Pontryagin dual as

$$\widehat{A} := \mathrm{Hom}(A, U(1)).$$

Example

- $\widehat{\mathbb{Z}} \simeq U(1)$ and $\widehat{U(1)} \simeq \mathbb{Z}$.
- For any positive integer n, one has $\widehat{\mathbb{Z}/n} \simeq \mathbb{Z}/n$. However, there is no canonical choice of such an isomorphism.

1. For any two Abelian groups A and B, we have $\widehat{A \times B} := \widehat{A} \times \widehat{B}$.
2. If A is finite, its Pontryagin dual factors through $\mathbb{Q}/\mathbb{Z} \subset U(1)$.

Therefore, Pontryagin duality for finite Abelian groups is determined by counting isomorphisms (for prime number p and positive integer k)

$$\widehat{\mathbb{Z}/p^k} \simeq \mathbb{Z}/p^k,$$

or equivalently, counting non-degenerate pairings

$$\mathbb{Z}/p^k \times \mathbb{Z}/p^k \to \mathbb{Q}/\mathbb{Z}.$$

In a on-going joint project with Mo Huang and Zhi-Hao Zhang, we would like to categorify this above notion of Pontryagin duality. So let us first recall basic definitions.

For us, a 2-group $\mathcal{G}$ is a monoidal category whose all morphisms are invertible, and all objects are invertible with respect to the monoidal product.

We denote the group of isomorphism classes of objects in $\mathcal{G}$ by $\pi_1(\mathcal{G})$ and the group of endomorphisms on the monoidal unit by $\pi_2(\mathcal{G})$. This notion is consistent with the topological approach, where one can deloop monoidal category $\mathcal{G}$ to get a 2-category $\mathrm{B}\mathcal{G}$ with one single object, and then its geometric realization is a pointed topological space with π_1 and π_2 agree with our above conventions.

A 2-group $\mathcal{G}$ is finite if we have both $\pi_1(\mathcal{G})$ and $\pi_2(\mathcal{G})$ to be finite groups.

A braided 2-group is a 2-group whose underlying monoidal category is equipped with a braiding.

A symmetric 2-group is a braided 2-group whose underlying braiding is symmetric. Symmetric 2-groups are categorications of Abelian groups. So we are going to define their Pontryagin duals as follows.

Pontryagin dual

Let $\mathcal{A}$ be a symmetric 2-group, we define its Pontryagin dual to be the category of braided functors

$$\widehat{\mathcal{A}} := \mathsf{Fun}_{br}(\mathcal{A}, \mathrm{B}U(1)).$$

It is equipped with a pointwise symmetric monoidal structure inherited from $\mathrm{B}U(1)$.

Example

- $\widehat{\mathrm{B}U(1)} \simeq \mathbb{Z}$, and $\widehat{\mathbb{Z}} \simeq \mathrm{B}U(1)$.
- $\widehat{U(1)} \simeq \mathrm{B}\mathbb{Z} \simeq U(1)$.
- $\widehat{\mathrm{B}\mathbb{Z}/n} \simeq \mathbb{Z}/n$, and $\widehat{\mathbb{Z}/n} \simeq \mathrm{B}\mathbb{Z}/n$ for any postive integer n.

By definition, for symmetric 2-groups $\mathcal{A}$ and $\mathcal{B}$, one has $\widehat{\mathcal{A} \times \mathcal{B}} \simeq \widehat{\mathcal{A}} \times \widehat{\mathcal{B}}$.

For finite symmetric 2-group $\mathcal{A}$, its Pontryagin dual factors through $\mathrm{B}\mathbb{Q}/\mathbb{Z}$.

Strictification Theorem (Joyal-Street;Johnson-Osorno)

Every symmetric 2-group is equivalent to one with trivial associators and unitors. In particular, such a strict symmetric 2-group $\mathcal{A}$ is determined by Abelian groups $\pi_1(\mathcal{A})$, $\pi_2(\mathcal{A})$ together with a group homomorphism $f : \pi_1(\mathcal{A}) \to \pi_2(\mathcal{A})$ corresponding to tensoring with the generator of $\pi_1(\mathbb{S}) \simeq \mathbb{Z}/2$.

Unlike finite Abelian groups, not every symmetric 2-group is Pontryagin self-dual. Using the above strictification theorem, we know that if a finite symmetric 2-group $\mathcal{A}$ is equivalent to its Pontryagin dual $\widehat{\mathcal{A}}$, then one must have

- $\pi_1(\mathcal{A}) \simeq \widehat{\pi_2(\mathcal{A})}$ and $\pi_2(\mathcal{A}) \simeq \widehat{\pi_1(\mathcal{A})}$,
- for a fixed choice of the above equivalence, we need $f : \pi_1(\mathcal{A}) \to \pi_2(\mathcal{A})$ to be identified with its dual $\widehat{f} : \widehat{\pi_2(\mathcal{A})} \to \widehat{\pi_1(\mathcal{A})}$.

Let us recall the Fourier transform on a finite Abelian group A. One can view it on different levels of abstraction.

On the level of objects

For any $\mathbb{C}$-valued function f on A, we define its Fourier transform to be a $\mathbb{C}$-valued function on its Pontryagin dual $\widehat{A}$:

$$\mathcal{F}(f) : \widehat{A} \to \mathbb{C}; \quad p \mapsto \sum_{x \in A} f(x) p(x).$$

Conversely, for any $\mathbb{C}$-valued function g on $\widehat{A}$, we define the inverse Fourier transform to be

$$\mathcal{F}^{-1}(g) : A \to \mathbb{C}; \quad x \mapsto \frac{1}{|G|} \sum_{p \in \widehat{A}} g(p) p(x).$$

More importantly, Fourier transform and its inverse are not only isomorphisms of vector spaces, but also isomorphisms between Hopf algebras

$$\mathrm{Fun}(A) \simeq \mathbb{C}[\widehat{A}], \qquad \mathbb{C}[A] \simeq \mathrm{Fun}(\widehat{A}),$$

which interchange pointwise products and convolution products.

On the level of vector spaces

For any finite dimensional A-representation (V, ρ_V), there is a canonical $\widehat{A}$-grading on its isotypical decomposition $V = \bigoplus_{p \in \widehat{A}} V_p$, where for any $p \in \widehat{A}$,

$$V_p := \{v \in V \mid \forall x \in A, \rho_V(x)(v) = p(x)v\}.$$

This induces an equivalence of symmetric monoidal categories

$$\mathrm{Rep}(A) \simeq \mathrm{Vect}_{\widehat{A}}.$$

On the level of categories

There is an equivalence of symmetric monoidal 2-categories $2\mathrm{Rep}(A) \simeq 2\mathrm{Rep}(\widehat{A})$.

1. Vect is an invertible bimodule category between Vect_A and $\mathrm{Rep}(A) \simeq \mathrm{Vect}_{\widehat{A}}$.
2. Given a finite semisimple category $\mathcal{C}$ with A-action, we construct its equivariantization

$$\mathcal{C}^A := \mathrm{Vect} \boxtimes_{\mathrm{Vect}_A} \mathcal{C},$$

   which inherits an $\widehat{A}$-action by (1).
3. Given a finite semisimple category $\mathcal{D}$ with $\widehat{A}$-action, we construct its equivariantization

$$\mathcal{D}^{\widehat{A}} := \mathrm{Vect} \boxtimes_{\mathrm{Vect}_{\widehat{A}}} \mathcal{D},$$

   which inherits an A-action by (1).

Finally, we would like to categorify the theory of Fourier transform. By comparison, we expect the follows.

On the level of objects

Let $\mathcal{A}$ be a finite symmetric 2-group.

- For a functor $f : \mathcal{A} \to \mathrm{Vect}$, we define its Fourier transform to be

$$\widehat{\mathcal{A}} \to \mathrm{Vect}; \quad p \mapsto \int_{x:\mathcal{A}} f(x) \otimes p(x).$$

- For a functor $g : \widehat{\mathcal{A}} \to \mathrm{Vect}$, we define the inverse Fourier transform to be

$$\mathcal{A} \to \mathrm{Vect}; \quad x \mapsto \int_{p:\widehat{\mathcal{A}}} g(p) \otimes p(x).$$

They provides an equivalence of categories $\mathrm{Fun}(\mathcal{A}, \mathrm{Vect}) \simeq \mathrm{Fun}(\widehat{\mathcal{A}}, \mathrm{Vect})$.

Fourier transform described above becomes an equivalence of symmetric monoidal categories, if

1. the LHS is equipped with pointwise product and the RHS is equipped with the convolution product;
2. the LHS is equipped with convolution product and the RHS is equipped with the pointwise product.

Moreover, depending on the symmetric monoidal structures, there are several symmetric co-monoidal structures such that altogether thery form Hopf algebra objects in 2Vect. Then the above equivalences preserve all these Hopf structures.

On the level of 1-categories

For any finite semisimple category $\mathcal{V}$ with $\rho_{\mathcal{V}} : \mathcal{A} \to \mathrm{End}(\mathcal{V})$, we consider its isotypical decomposition

$$\mathcal{V} \simeq \int_{p:\widehat{A}} \mathcal{V}_p,$$

where $\mathcal{V}_p$ consists of pairs (v, ϕ^v): v is an object in $\mathcal{V}$ and $(\phi^v)_x : \rho_{\mathcal{V}}(x)(v) \to p(x) \otimes v$ is a natural isomorphism with respect to x in $\mathcal{A}$.

Viewed as an $\widehat{\mathcal{A}}$-grading on $\mathcal{V}$, this provides an equivalence of 2-categories

$$2\mathrm{Rep}(\mathcal{A}) \simeq 2\mathrm{Vect}_{\widehat{\mathcal{A}}}.$$

Moreover, this equivalence preserves symmetric monoidal structures: the LHS is equipped with pointwise product while the RHS is equipped with convolution product.

I would like to remind the audience that, unlike Abelian groups, the same underlying 2-group could have several different symmetric monoidal structures. Therefore, the equivalence seems paradoxical at first: $2\mathsf{Rep}(\mathcal{A}) \simeq 2\mathsf{Vect}_{\widehat{\mathcal{A}}}$, since the LHS doesn't depend on the symmetric monoidal structures of $\mathcal{A}$ by definition, but the RHS obviously depends on the symmetric monoidal structures.

The reason is that, even though $\mathcal{A}$ could have several different symmetric monoidal structures which lead to potentially different Pontryagin duals $\widehat{\mathcal{A}}$, they have to become equivalent after linearization.

Interpreted in another way, this warns us that to determine $\widehat{\mathcal{A}}$, it is not enough to know all the invertible objects in $2\mathsf{Rep}(\mathcal{A})$, which turns out to be $\mathsf{Fun}_{\otimes}(\mathcal{A}, \mathrm{B}U(1))$. In addition, we need to specify a section of group homomorphism that goes from the invertible objects in $2\mathsf{Rep}(\mathcal{A})$ to the group of connected components in $2\mathsf{Rep}(\mathcal{A})$.

On the level of 2-categories

There is an equivalence of symmetric 3-categories $3\mathrm{Rep}(\mathcal{A}) \simeq 3\mathrm{Rep}(\widehat{\mathcal{A}})$.

1. 2Vect is an invertible bimodule 2-category between $2\mathrm{Vect}_{\mathcal{A}}$ and $2\mathrm{Vect}_{\widehat{\mathcal{A}}}$.
2. Given a finite semisimple 2-category $\mathfrak{C}$ with $\mathcal{A}$-action, we construct its equivariantization
$$\mathfrak{C}^{\mathcal{A}} := 2\mathrm{Vect} \boxtimes_{2\mathrm{Vect}_{\mathcal{A}}} \mathfrak{C},$$
which inherits an $\widehat{\mathcal{A}}$-action by (1).
3. Given a finite semisimple 2-category $\mathfrak{D}$ with $\widehat{\mathcal{A}}$-action, we construct its equivariantization
$$\mathfrak{D}^{\widehat{\mathcal{A}}} := 2\mathrm{Vect} \boxtimes_{2\mathrm{Vect}_{\widehat{\mathcal{A}}}} \mathfrak{D},$$
which inherits an $\mathcal{A}$-action by (1).

For outlooks,

1. We would like to apply this construction to understand the fusion rules of 2-representations of general 2-groups, or even better, to understand the fusion rules of general fusion 2-categories.
2. Pontryagin self duality and Fourier transform also plays an important role in the study of Tambara-Yamagami categories. It would be interesting to generalize this to 2-groups and corresponding lattice models.
3. Pontryagin duality and Fourier transform can be defined for non-finite symmetric 2-groups. Could we find a framework that extends our discussion of fusion 2-categories?