Anti-Self-Dual Yang-Mills eq.

4-dimensional Wess-Zumino-Witten Models (WZW) and unification of integrable systems

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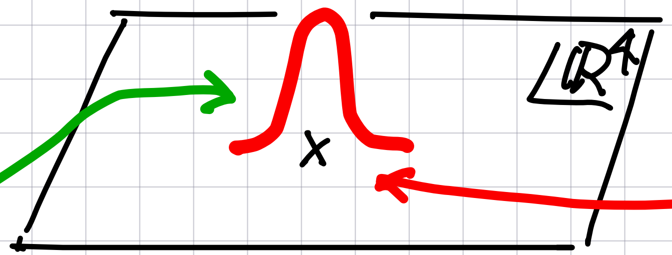
- MH, Shan-Chi Huang, Hiroaki Kanno, 2212.11800 Prog. Theor. Exp. Phys. (PTEP) 2023-4, 043B03 ; etc.
- MH, S.C. Huang, OCNMP (受理), 2408.16554

§1 Introduction

Anti-Self-Dual (ASD) Yang-Mills (YM) eqs.

★ Instantons : special "soliton" solutions

action density $\mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$



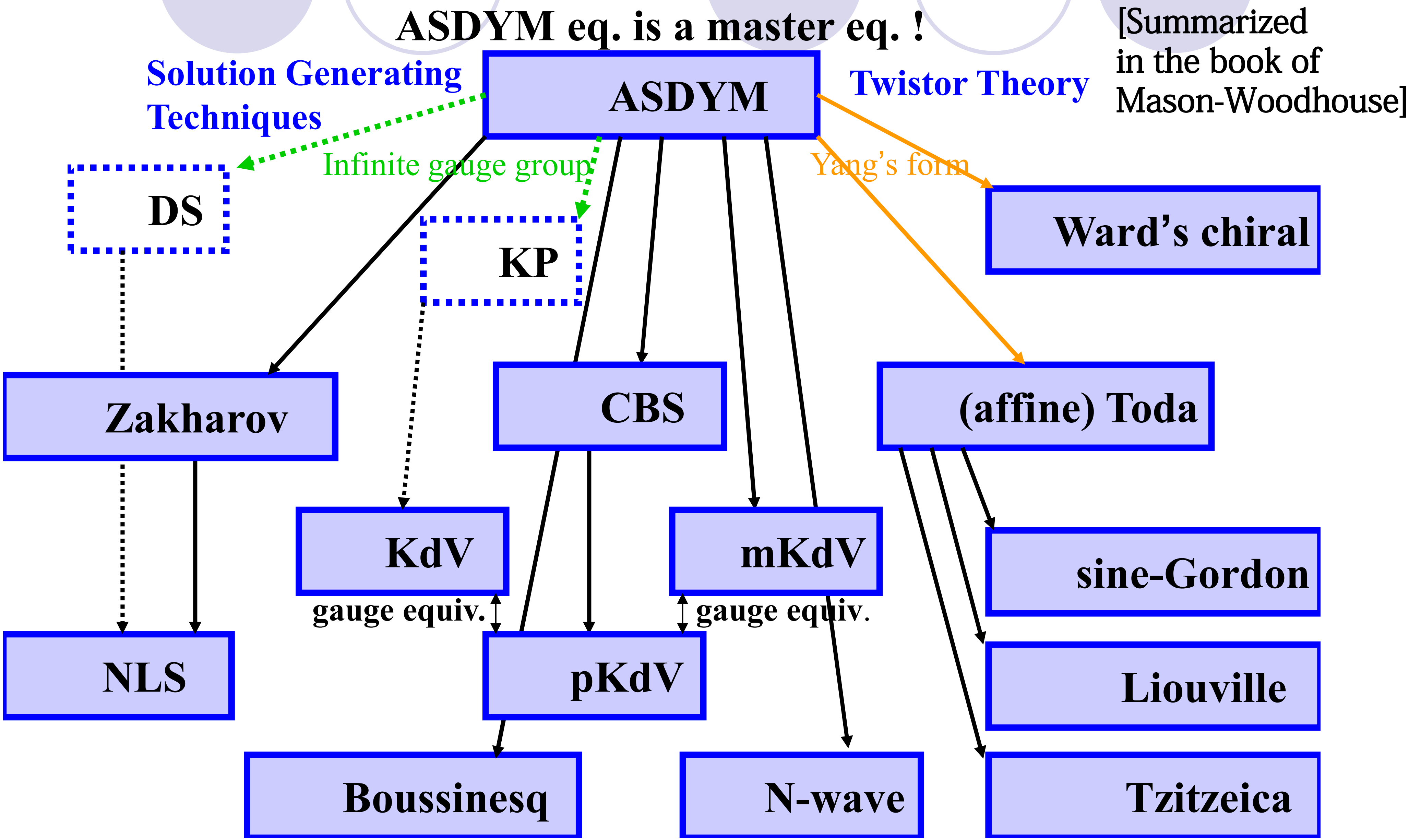
localized at a point

- reveal non-perturbative aspects of QFT
- Application to geometry

  e.g. Donaldson inv., Nekrasov part. fcn., ⋯

★ "master eq" of integrable systems [R. Ward, ⋯]

Ward's conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.



ASDYM eq. (in 4 dim, $G_{YM} = GL(N, \mathbb{C})$ or subgp.)

G_{YM} ← gauge group

$$\underset{\text{Hodge dual}}{*} F_{\mu\nu} = -F_{\mu\nu} \qquad \underbrace{F_{\mu\nu}}_{\text{field strength}} = \partial_\mu \underbrace{A_\nu}_{\substack{\text{gauge field} \\ N\times N}} - \partial_\nu A_\mu + \underset{\substack{\uparrow \\ \text{commutator}}}{[A_\mu, A_\nu]}$$

$\mu, \nu \in \{1, 2, 3, 4\}$

$\Leftrightarrow \quad F_{12} = -F_{34}, \quad F_{13} = -F_{42}, \quad F_{14} = -F_{23}$

later

$\Leftrightarrow \quad F_{zw} = 0, \quad F_{\tilde{z}\tilde{w}} = 0, \quad F_{z\tilde{z}} + F_{w\tilde{w}} = 0 \qquad$ (in $\mathbb{E}$)

↓ rewrite

$\Leftrightarrow \quad \partial_{\tilde{z}}\left((\partial_z a) a^{-1}\right) + \partial_{\tilde{w}}\left((\partial_w a) a^{-1}\right) = 0 \qquad$ Yang's eq.

Reduction to NLS from ASDYM $(G=SU(2))$

ASDYM : $F_{zw}=0,\ F_{\tilde{z}\tilde{w}}=0,\ F_{z\tilde{z}}-F_{w\tilde{w}}=0$

① $\partial_w-\partial_{\tilde{w}}=0,\ \partial_{\tilde{z}}=0$ (dim. reduction)

② $A_{\tilde{w}}=0,\ A_{\tilde{z}}=\frac{i}{2}\begin{pmatrix}-1&0\\0&1\end{pmatrix},\ A_w=\begin{pmatrix}0&-\psi\\\bar{\psi}&0\end{pmatrix}$

$$A_z=i\begin{pmatrix}\psi\bar{\psi}&-\psi_x\\-\bar{\psi}_x&-\bar{\psi}\psi\end{pmatrix}\qquad \psi=\psi(z,\underset{\sim}{x}),\ \ x=\underset{\sim}{w+\tilde{w}},\qquad \psi_x=\partial_x\psi$$

$i\underset{t}{\psi_z}+\psi_{xx}-2\psi\bar{\psi}\psi=0$: **NLS eq.!** [Mason-Sparling, PLA ('89) 137, 29]

(t,x) are real $\Rightarrow$ not $(++++)$ but $(++--)$

A Unified theory of integrable systems

EoM = ASDYM eq. !

4d CS

↓ [Costello-Yamazaki (-Witten)]

various [Delduc-Lacroix-Magro-Vicedo], solvable models [Yoshida (K), Sakamoto, Fukushima, ⋯] ⋯

(spin chains, PCM, ⋯)

4d WZW (++−−)

[Ward] ↓ [Mason-Woodhouse]

various integrable eqs.

(KdV, NLS, Toda, ⋯)

A Unified theory of integrable systems

6d meromorphic
Chern-Simons (CS)

[Costello]
[Bittleston-Skinner]

4d CS ← duality? → 4d WZW (++−−)

[Costello-Yamazaki (-Witten)]

[Ward] [Mason-Woodhouse]

various solvable models [Delduc-Lacroix-Magro-Vicedo], [Yoshida (K), Sakamoto, Fukushima, …] …
(spin chains, PCM, …)

various integrable eqs.
(KdV NLS, Toda, …)

Plan of Talk (simple discussion)

§1 Introduction (12 min)

§2 4d WZW model (8 min)

§3 Soliton Solutions of ASDYM (10 min)

§4 Conclusion & Discussion (7 min)

(I'm a physicist)

§2. 4dim Wess-Zumino-Witten (WZW) model

[Donaldson '85]
[Losev-Moore-Nekrasov -Shatashvili '96]

4-dim WZW (4dWZW) model

- analogue of 2-dim WZW model [Inami-Kanno-Ueno-Xiong '96] ...
- EOM = Yang's eq ≡ Anti-Self-Dual Yang-Mills eq. (ASD)
- In the split signature $(+,+,-,-)$, SFT action of $N=2$ string theory ['91 [Ooguri-Vafa]]

← Today we focus on

We discuss classical soliton sols. of it

↑ implication application

Action: $S_{WZW_4} = S_\sigma + S_{WZ}$

$\sigma(x) \in G$
↑
NOT G_{YM}

$$S_\sigma = \frac{i}{4\pi}\int_{M_4} \omega \wedge \mathrm{Tr}\left[(\partial\sigma)\sigma^{-1} \wedge (\tilde{\partial}\sigma)\sigma^{-1}\right]$$

$$S_{WZ} = -\frac{i}{12\pi}\int_{M_4} A \wedge \mathrm{Tr}\left[(d\sigma)\sigma^{-1}\right]^3$$

$(z, w, \tilde{z}, \tilde{w})$: local coords of M_4

w/ $\omega = dA$: Kähler form of M_4

M_4 : flat 4-dim space-time $\quad \omega = \frac{i}{2}(dz\wedge d\tilde{z} - dw\wedge d\tilde{w})$

$$d = \partial + \tilde{\partial}, \quad \partial = dw\,\partial_w + dz\,\partial_z, \quad \tilde{\partial} = d\tilde{w}\,\partial_{\tilde{w}} + d\tilde{z}\,\partial_{\tilde{z}}$$

EoM: $\tilde{\partial}\left(\omega \wedge (\partial\sigma)\sigma^{-1}\right) = 0 \iff$ Yang's eq.

⇕

ASDYM eq. !

N=2 string theory

# WS SUSY	Name	Target sp.	field contents
$N=0$	Bosonic String	$(1+25)$ dim	$g_{\mu\nu}, B_{\mu\nu}, \phi, \cdots$
$N=1$	Superstring	$(1+9)$ dim	〃 〃
$N=2$	$N=2$ string	$(2+2)$ dim	massless scalar only!

open N=2 string

$\sigma = e^{\varphi}$ ← the massless scalar

[Ooguri-Vafa, '91]

$$S_{WZW_4} = \underbrace{(\text{in terms of } \varphi)}_{\substack{\parallel \\ S_{N=2\text{ string}} \\ (SFT)}} \rightsquigarrow \text{n-pt. fcn of } \varphi$$

(coincides with WS calculations)

N=2 string theory

<u>Rmk</u> Xianghang Zhang (Nagoya) studies homotopy algebra formulation of $S_{N=2\text{ string}}$ (finally, we comment again)

open N=2 string

$\sigma = e^{\varphi}$ ← the massless scalar [Ooguri-Vafa,'91]

$S_{WZW_4} = \underbrace{(\text{in terms of } \varphi)}_{\substack{\rotatebox{90}{=} \\ S_{N=2\text{ string}}\ (\mathrm{SFT})}} \rightsquigarrow$ n-pt. fcn of φ (coincides with WS calculations)

§3. Soliton Solutions of ASDYM eq.

Yang's eq. (on $\mathbb{C}^4$: complexified space-time)

$$\partial_{\tilde{z}}\left((\partial_z \sigma)\sigma^{-1}\right) - \partial_{\tilde{w}}\left((\partial_w \sigma)\sigma^{-1}\right) = 0$$

$\sigma \in G = GL(N;\mathbb{C})$

※ Real slice

$(z, w, \tilde{z}, \tilde{w}) \in \mathbb{C}^4$, $\quad ds^2 = dz\,d\tilde{z} - dw\,d\tilde{w}$

① $\left[\begin{array}{l} z = x^1 + x^3,\ w = x^2 + x^4 \\ \tilde{z} = x^1 - x^3,\ \tilde{w} = x^4 - x^2 \end{array}\right]$ $\downarrow$

$\mathbb{R}^4(+,+,-,-)$

Ultrahyperbolic sp. $\mathbb{U}$

TODAY!

② $z = x^1 + i x^2,\ w = x^3 + i x^4$, $\tilde{z} = \bar{z},\ \tilde{w} = -\bar{w}$ $\downarrow$

$\mathbb{R}^4(++++)$

Euclid sp. $\mathbb{E}$

Lax representation :

$N\times N$ const matrix

$$(*)\begin{cases} Lf = \sigma\,\partial_w(\sigma^{-1}f) - (\partial_{\tilde z} f)\,\zeta = 0 & \begin{pmatrix}\text{right}\\ \text{action}\end{pmatrix} \\ Mf = \sigma\,\partial_z(\sigma^{-1}f) - (\partial_{\tilde w} f)\,\zeta = 0 \end{cases}$$

compatible condition $\Rightarrow$ Yang's eq.

$L(M\phi) - M(L\phi) = 0$

Darboux trf.

[Nimmo-Gilson-Ohta '00] [Gilson-H-Huang-Nimmo '20]

$$(D)\begin{cases} \tilde f = f\zeta - \theta\Lambda\theta^{-1}f \\ \tilde\sigma = -\theta\Lambda\theta^{-1}\sigma \end{cases}$$

$\underset{N\times N}{\theta}$: special sol. for Λ

special value

Under the Darboux trf. (*) is form invariant $\left(\text{i.e. } \begin{matrix}\tilde L\tilde f = 0\\ \tilde M\tilde f = 0\end{matrix}\right.$

n-iterations of (D) from a trivial seed sol.

$(\sigma = 1)$

$$\underset{N\times N}{\sigma_n} = \begin{vmatrix} \theta_1 & \cdots & \theta_n & 1 \\ \theta_1^{(1)} & \cdots & \theta_n^{(1)} & 0 \\ \vdots & & \vdots & \vdots \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 0 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

(θ_1: $N\times N$; 1: $N\times N$)

$$\theta_k^{(l)} := \theta_k \Lambda_k^l$$

$$(\theta_i, \Lambda_i): \quad \partial_w \theta_i = \partial_{\tilde{z}} \theta_i \Lambda_i$$

$$\partial_z \theta_i = \partial_{\tilde{w}} \theta_i \Lambda_i$$

Wronskian-type!

Quasideterminant

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} := \underset{N\times N}{d} - C A^{-1} B \quad \text{(Schur complement)}$$

squares

n-soliton sols. for $G = SL(2,\mathbb{C})$:

[H-Huang. '20]

$$\sigma_n = \begin{vmatrix} \theta_1 & \cdots & \theta_n & 1 \\ \theta_1^{(1)} & \cdots & \theta_n^{(1)} & 0 \\ \vdots & & \vdots & \vdots \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 0 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$\theta_k = \begin{pmatrix} e^{L_k} & e^{-\bar{L}_k} \\ -e^{-L_k} & e^{\bar{L}_k} \end{pmatrix}, \quad \Lambda_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \mu_k \end{pmatrix}$$

$$L_k = \lambda_k \alpha_k z + \beta_j \tilde{z} + \lambda_j \beta_j w + \alpha_j \tilde{w}$$

(linear in space-time coord)

<u>Rmk</u> (U) $\mu_k = \bar{\lambda}_k, \ |\mu_k| = 1$

$\Rightarrow G = SU(2)$

(E) $\mu_k = -1/\bar{\lambda}_k, \ |\mu_k| = 1$

$\Rightarrow G = U(2)$

Non-abelian system

&

Calculate the WZW action density of them

One soliton (on $\mathbb{U}$)

※ $\lambda = \bar{\lambda} \Rightarrow \mathcal{L}_\sigma \equiv 0$

$$\sigma = -\theta\Lambda\theta^{-1}, \quad \theta = \begin{pmatrix} e^{L} & e^{-\bar{L}} \\ -e^{-L} & e^{\bar{L}} \end{pmatrix}, \quad \Lambda = \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}}$$

$\operatorname{sech} x = \dfrac{1}{\cosh x}$

$\Downarrow$

$$\mathcal{L}_\sigma = \frac{1}{8\pi}\, \underbrace{d_{11}}\, \operatorname{sech}^2 X, \qquad d_{11} \propto \underbrace{(\lambda-\bar{\lambda})^3}$$

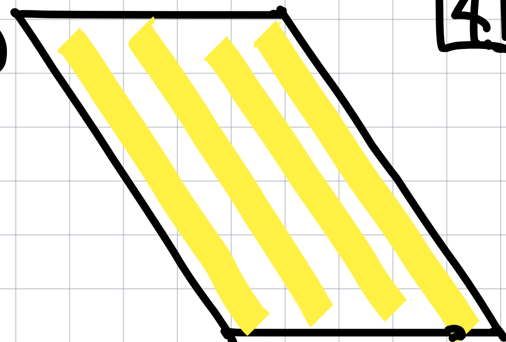
$X := L + \bar{L}$: linear in x^μ

$\mathcal{L}_{WZ} \equiv 0$ (identically)

peak

$X = 0$

4D



3-dim hyperplane (codim 1)

not instanton!

Similar!

cf. KP soliton

$$u = 2\partial_x^2 \log\left(e^{X} + e^{-X}\right) \propto \operatorname{sech}^2 \underbrace{X}$$

linear in t, x, y

Two Soliton ($\mathcal{L}_\sigma$)

$X_k = L_k + \bar{L}_k,\quad \Theta_{12} = \Theta_1 - \Theta_2$

$i\Theta_k = L_k - \bar{L}_k$

$$\mathcal{L}_\sigma = \frac{\left[A\cosh^2 X_1 + B\cosh^2 X_2 + C_\pm \cosh^2\left(\frac{X_1 + X_2 \pm i\Theta_{12}}{2}\right) + D_\pm \cosh^2\left(\frac{X_1 - X_2 \pm i\Theta_{12}}{2}\right)\right]}{2\pi\left(a\cosh(X_1+X_2) + b\cosh(X_1-X_2) + c\cos\Theta_{12}\right)^2}$$

non-singular

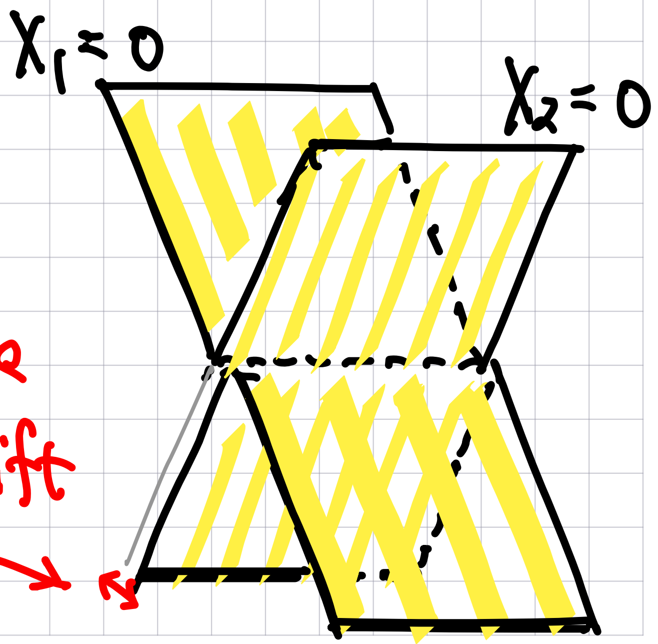
$\xrightarrow[X_1:\,\mathrm{const}]{r\to\infty} \propto \mathrm{sech}^2(X_1 \pm \delta_1)$

$\xrightarrow[X_2:\,\mathrm{const}]{r\to\infty} \propto \mathrm{sech}^2(X_2 \pm \delta_2)$

$\xrightarrow[\text{otherwise}]{r\to\infty} 0$

phase shift

(non-linear effect)

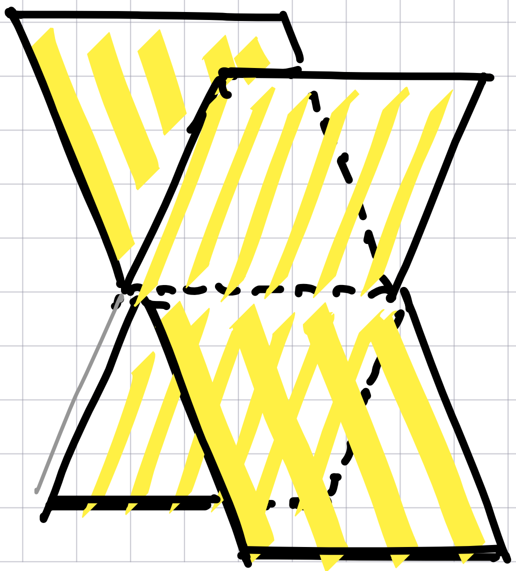
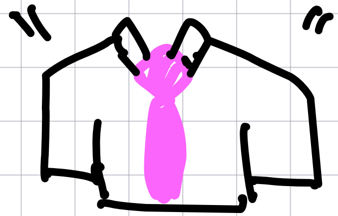


Two Soliton

$\mathcal{L}_{WZ}$ = (very long many terms) non-singular

$\xrightarrow{r\to\infty}$ 0 (in any direction)

$\mathcal{L}_{total} = \mathcal{L}_{\alpha}$ + ("dressing" in the middle region)

=  + 

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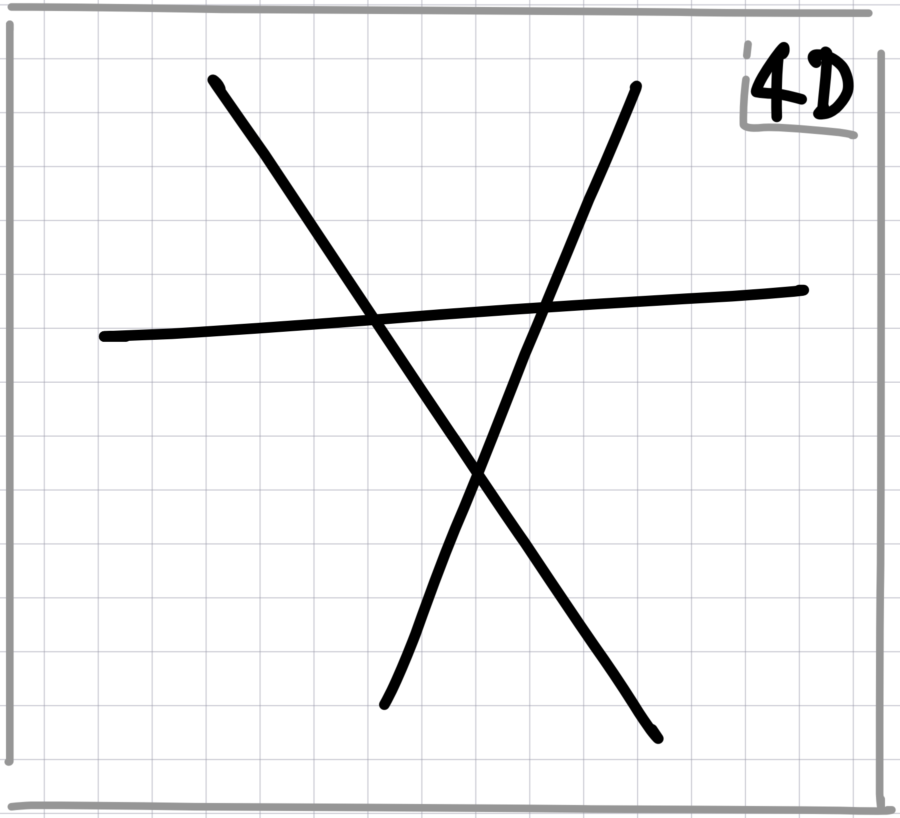
n- soliton sol. = "non-linear superposition of n one solitons [H-Huang '22]



intersecting n hyperplanes (with phase shifts)

Rmk 1 Reduction to (1+2) dim.

Consider $(x^1, x^2, x^3, x^4) \longrightarrow (\overset{\overset{t\ (\text{time})}{\shortparallel}}{x^1}, x^3, x^4)$

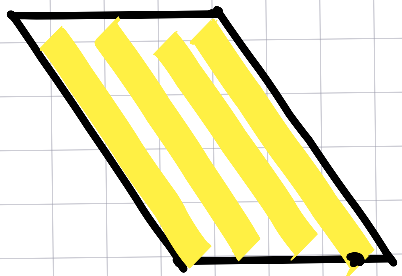
The soliton sol. $\sigma(\alpha_k = \lambda_k \beta_k)$ solves EoM in (1+2)d

$\Updownarrow$

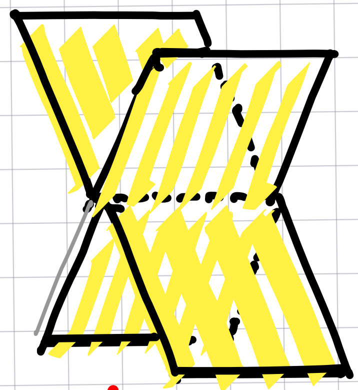
$\because\ L_k = (\lambda_k \alpha_k + \beta_k)x^1 + \underbrace{(\lambda_k \beta_k - \alpha_k)}_{=0} x^2 + \cdots \quad \blacksquare$

Hamiltonian $\mathcal{H} = \sum_{i=1}^{3} \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_i)} \partial_t \phi_i - \mathcal{L}$ $\quad (\mathcal{H}_{WZ} \equiv 0\ !)$

One soliton



Two soliton (no dressing)



Energy density has the same peaks as action density.

Rmk 2 Euclidean case $\mathbb{E}$

The soliton sols. : almost the same as in $\mathbb{U}$

Instanton solution (well-known in YM)

(Ex) $G_{YM} = SU(2)$ 't Hooft 1-instanton

$$\mathcal{L}_a \propto \frac{(z\bar{z}+w\bar{w})^3}{\underbrace{(z\bar{z}w\bar{w})}_{}{}^2(1+z\bar{z}+w\bar{w})^2}$$

$$\mathcal{L}_{wz} \propto \frac{(z\bar{z}+w\bar{w})(z\bar{z}-w\bar{w})^2}{\underbrace{(z\bar{z}w\bar{w})}_{}{}^2(1+z\bar{z}+w\bar{w})^4}$$

localized at the origin (codim 4)

singular $\rightsquigarrow$ resolved in NC spaces?

§4 Conclusion and Discussion

We constructed new-type of codim 1 solitons and calculated action densities of WZW_4 model.

$\rightsquigarrow$ intersecting codim 1 branes in the $N=2$ string
(new branes)

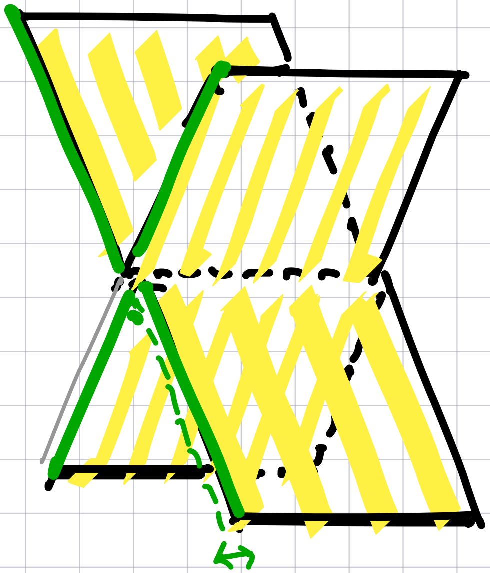
Future works:

① Classification of the soliton sols.

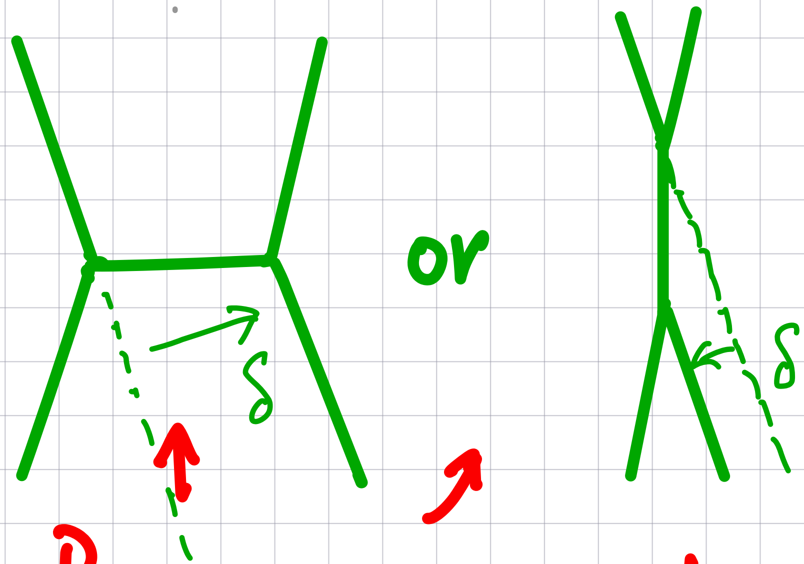
② Quantization of integrable systems

③ Non-Commutative (NC) extension of them

① Classification

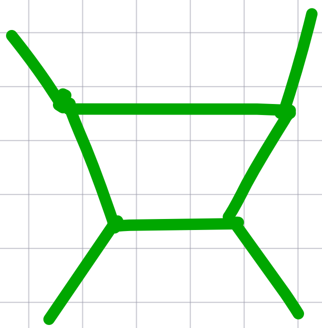


$\Leftrightarrow$



Resonances : origins of phase shifts

KP solitons



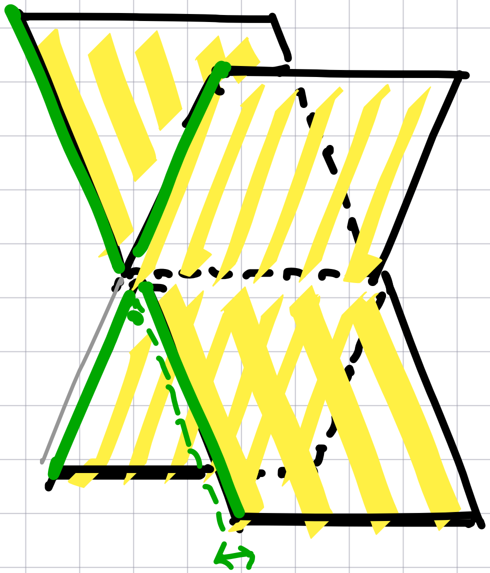
$\longleftrightarrow$ positive Grassmannians

soliton webs

[Kodama-Williams, Invent. Math (2014)]

① Classification

work in progress with S.C. Huang (名古屋)
& Shangshuai Li and Da-jun Zhang (上海)



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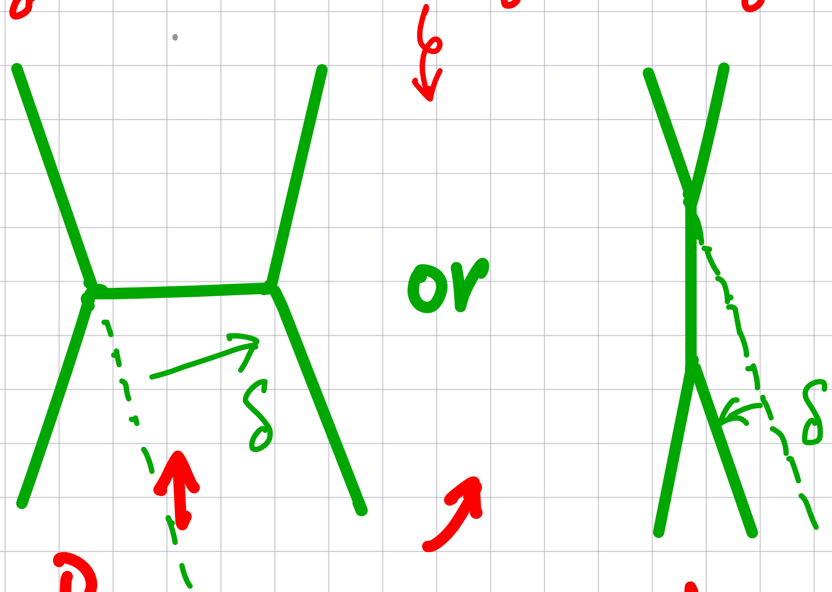
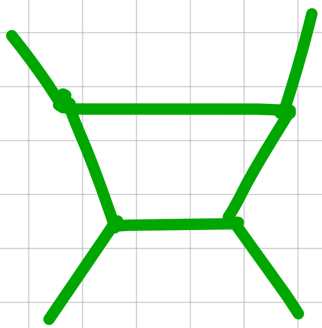


Resonances : origins of phase shifts

WZW$_4$ solitons



soliton planes

$\leftrightarrow$?? $\rightsquigarrow$ moduli sp. of solitons

$\rightsquigarrow$ new invariants in geometry?

A Unified theory of integrable systems

6d meromorphic Chern-Simons (CS)

↙ ↘

4d CS ← duality? → 4d WZW (++−−)

↓ ↓

various solvable models (spin chains, PCM, ⋯)

various integrable eqs. (KdV NLS, Toda, ⋯)

A Unified theory of integrable systems

6d meromorphic Chern-Simons (CS)

↙ ↘

4d CS

↓

various solvable models (spin chains, PCM, ⋯)

SFT action = 4d WZW (++−−)

⇓ Quantization

vanishing n(≥4)-pt. fcn via homotopy alg formulation [Xianghang Zhang]

↓

various integrable eqs. (KdV NLS, Toda, ⋯) [cf. Yuji Okawa]

② Quantization

homotopy alg. formulation of Lagrangian multiforms

← [Nijhoff, Suris, ...]

6d meromorphic Chern-Simons (CS)

[H-Huang, arXiv: 2408.16554]

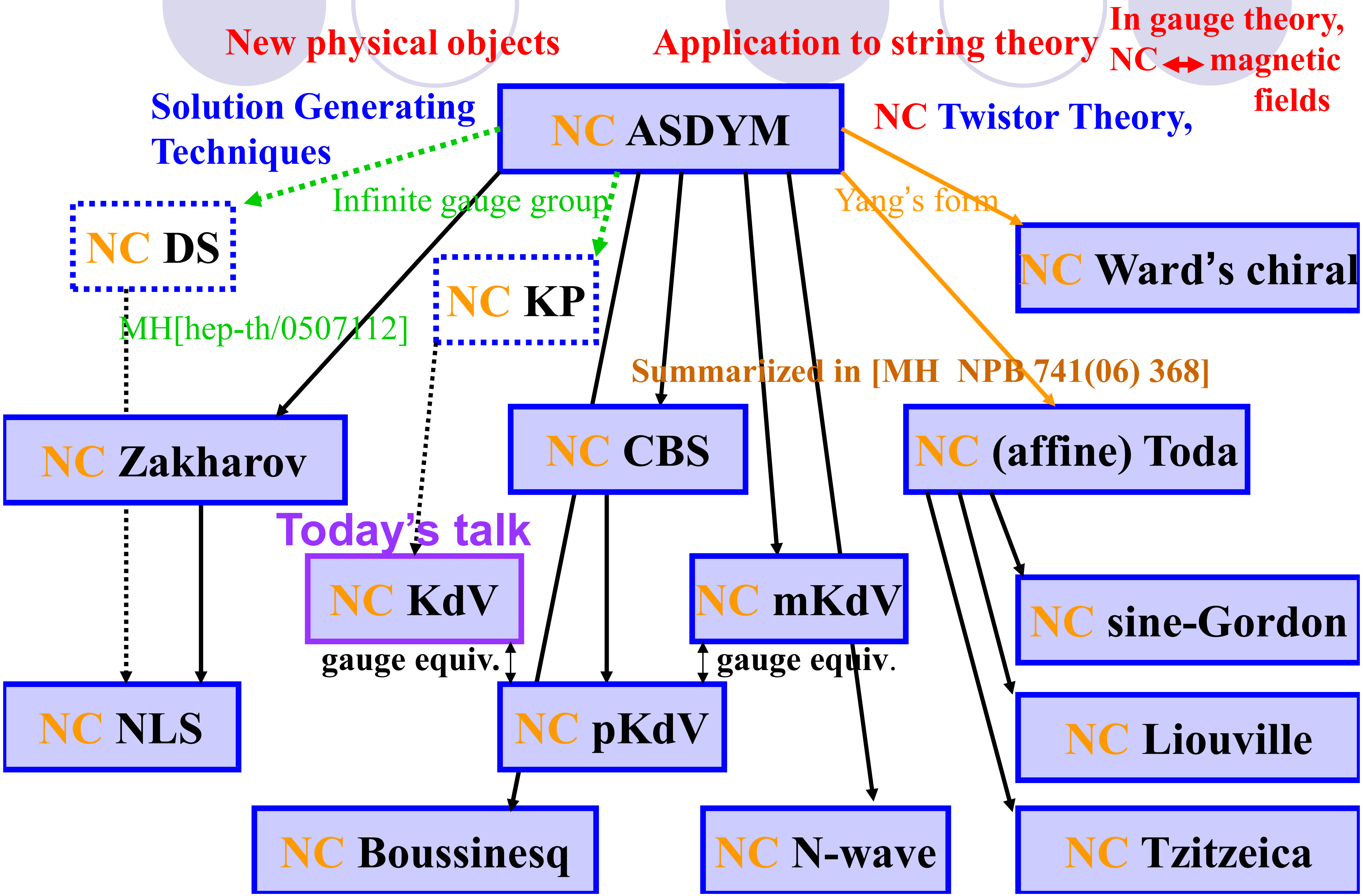
↙	↘
4d CS	4d WZW (++−−)
↓	↓
various solvable models	various integrable eqs.

Systematic Quantization for Integrability

(anomaly, YBE, S-matrix fact, ...)

NC Ward's conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.

MH &K.Toda, PLA316 (03)77 [hepth/0211148]



Reduction to NLS from ASDYM $(G = U(2))$

NC NC

U(1) part is crucial

ASDYM : $F_{zw} = 0,\ F_{\tilde{z}\tilde{w}} = 0,\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0$

① $\partial_w - \partial_{\tilde{w}} = 0,\ \partial_{\tilde{z}} = 0$ (dim. reduction)

② $A_{\tilde{w}} = 0,\ A_{\tilde{z}} = \frac{i}{2}\begin{pmatrix} -1 & \\ & 1 \end{pmatrix},\ A_w = \begin{pmatrix} 0 & -\psi \\ \bar{\psi} & 0 \end{pmatrix}$

NOT traceless

$$A_z = i\begin{pmatrix} \psi\bar{\psi} & -\psi_x \\ -\bar{\psi}_x & -\bar{\psi}\psi \end{pmatrix} \qquad \psi = \psi(z, x^{w+\tilde{w}}),\quad \psi_x = \partial_x \psi$$

$i\psi_z + \psi_{xx} - 2\psi\bar{\psi}\psi = 0$: NC NLS eq. !

this ordering is important

A Unified theory of NC integrable systems

③

6d meromorphic
NC Chern-Simons (CS)

NC
4d CS

↓

various NC
solvable models
(spin chains, PCM, ⋯)

← duality? →
key: Quasideterminant?

NC
4d WZW (+ + - -)

↓

various NC
integrable eqs.
(KdV, NLS, Toda, ⋯)

Thank You Very Much

謝謝