Introduction to the tetrahedron equation

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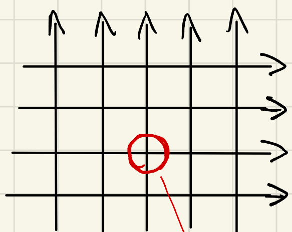
§ 1 2D integrable lattice model and Yang-Baxter equation L1

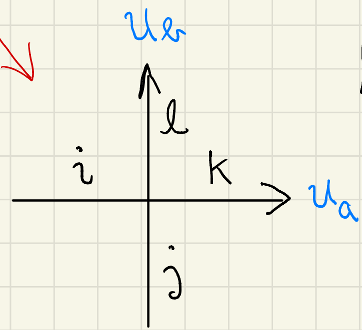


2-dim. (square) lattice

To each edge we assign a "spin"

$i \in \{1, 2, \dots, N\}$ $\dim V = N$

$$R_{ab}(u_a, u_b) \in \mathrm{End}\big(V_a(u_a) \otimes V_b(u_b)\big)$$

$E_{ij}^{kl}(u_a, u_b) =$ 

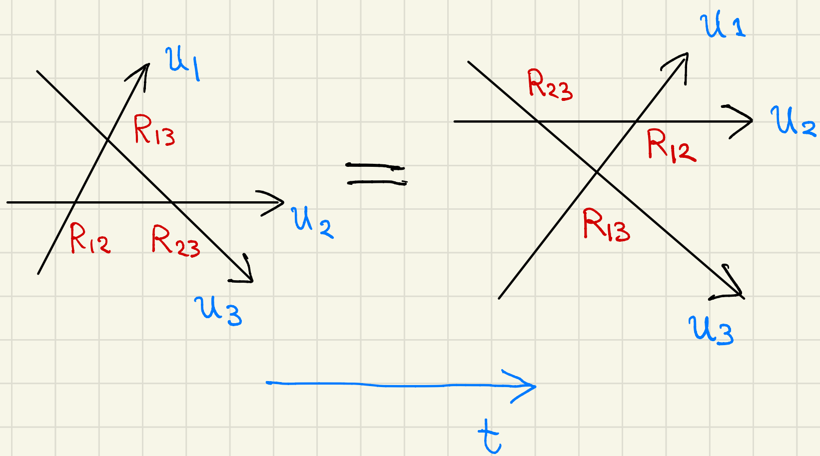
$$(R_{ab})_{ij}^{kl} := e^{-\beta E_{ij}^{kl}(u_a, u_b)}$$

R-matrix = the Boltzmann weight

"vertex model"

If the R-matrix satisfies the Yang-Baxter equation

$$R_{23}\, R_{13}\, R_{12} = R_{12}\, R_{13}\, R_{23} \in \mathrm{End}\,(V_1 \otimes V_2 \otimes V_3),$$



$$R_{23}\,(R_{13}\,R_{12})\,R_{23}^{-1} \;\text{"="}\; R_{12}\,R_{13}$$

By the adjoint action of R_{23} we can exchange R_{12} and R_{13}.

the vertex model is integrable.

$\exists$ Infinite number of mutually commuting Hamiltonians H_n

$n = 0, 1, 2, \cdots\cdots$

The symmetry undelying the art of the vertex model is the quantum affine algebra $A = U_q(\widehat{sl_N})$.

Coproduct $\Delta : A \longrightarrow A \otimes A$

$$P(x \otimes y) = y \otimes x$$

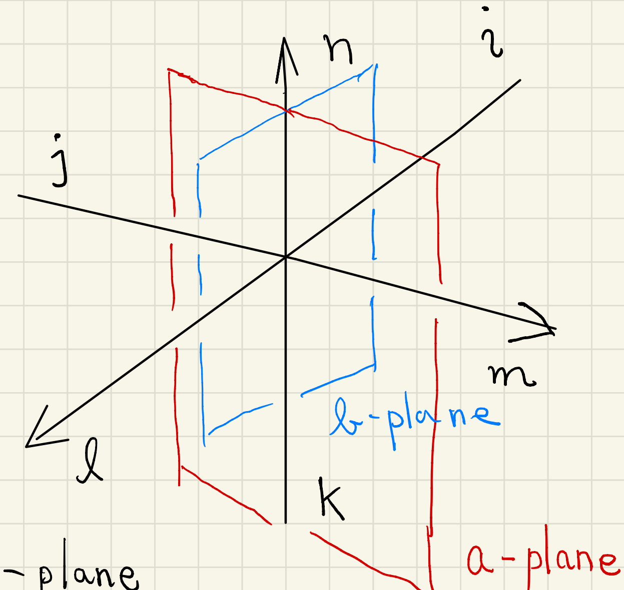
$R \Delta = \Delta' R$ R-matrix as the intertwiner of Δ and $\Delta' = P\Delta$

§ 2 3D generalization (Tetrahedron equation)

2D (square) lattice $\longrightarrow$ 3D (cubic) lattice

$$R_{abc} \in \mathrm{End}(V_{bc} \otimes V_{ca} \otimes V_{ab})$$

$$(R_{abc})^{lmn}_{ijk} =$$



$V_{ab} \Leftrightarrow$ intersection of a-plane and b-plane

$\mathcal{F} = \bigoplus \mathbb{C}\,|m\rangle$ a vector space with a basis $\{|m\rangle\}_{m\in\mathbb{Z}}$ L5

$$R \in \mathrm{End}(\mathcal{F}^{\otimes 3})$$

$$R\big(|i\rangle \otimes |j\rangle \otimes |k\rangle\big) = \sum_{l,m,n} R_{ijk}^{lmn} \big(|l\rangle \otimes |m\rangle \otimes |n\rangle\big)$$

$$\Rightarrow \sum_{l,m} |l\rangle \otimes |m\rangle \otimes R_{ij}^{lm}\,|k\rangle$$

$R_{ij}^{lm} \in \mathrm{End}(\mathcal{F})$ $\qquad R_{ij}^{lm}\,|k\rangle = \sum_{n} R_{ijk}^{lmn}\,|n\rangle$

$\rightsquigarrow$ A vertex model with an $\mathrm{End}(\mathcal{F})$-valued Boltzmann weight R_{ij}^{lm}

the tetrahedron equation

A.B. Zamolodchikov (1980)

$$R_{124} R_{135} R_{236} R_{456} = R_{456} R_{236} R_{135} R_{124}$$

$$\in \mathrm{End}(\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3 \otimes \mathcal{F}_4 \otimes \mathcal{F}_5 \otimes \mathcal{F}_6)$$

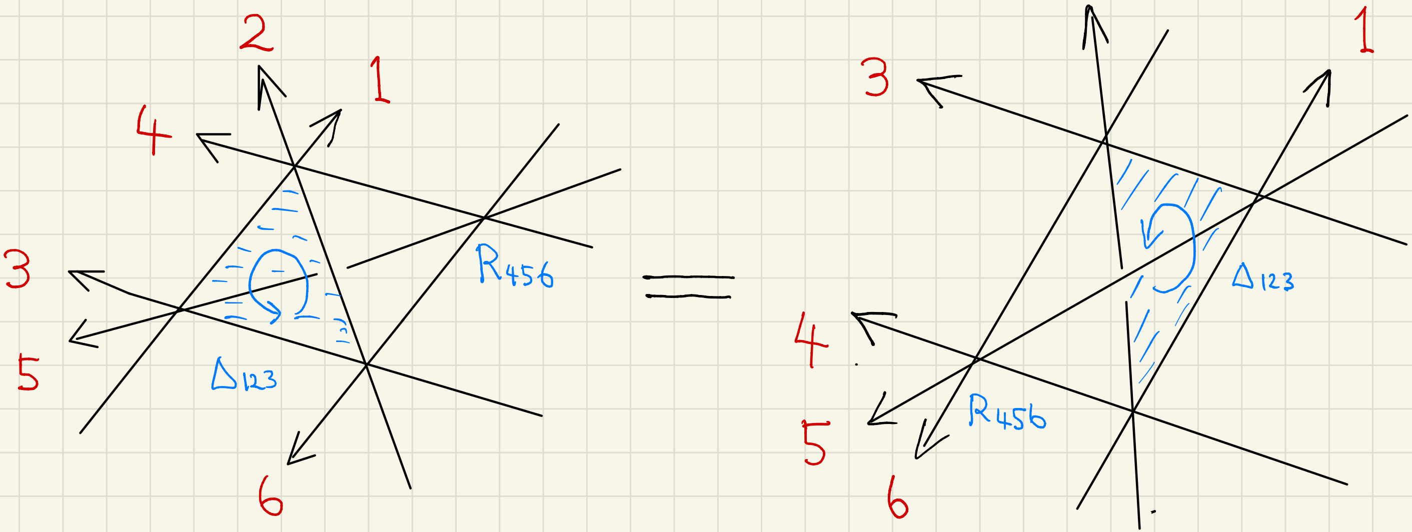
$$R_{124} \to \hat{R}_{12} \qquad R_{135} \to \hat{R}_{13} \qquad R_{236} \longrightarrow \hat{R}_{23}$$

$\hat{R}_{12} \hat{R}_{13} \hat{R}_{23} \cong \hat{R}_{23} \hat{R}_{13} \hat{R}_{12}$ up to the conjugation by R_{456}

This may be regarded as a quantization of Yang-Baxter eq.

along the direction of the auxiliary sp. $\mathcal{F}_4 \otimes \mathcal{F}_5 \otimes \mathcal{F}_6$

Graphical representation



$$R_{124}\,R_{135}\,R_{236}\,R_{456} = R_{456}\,R_{236}\,R_{135}\,R_{124}$$

Dual description : assign R_{abc} to each face

vertex formalism v.s. face formalism

§3 Solution to the tetrahedron eq.

The tetrahedron eq. is a highly overdetermined system

$$\dim \mathcal{F} = N \Rightarrow N^{12} \text{ conditions for } N^{6}\text{-components}$$

We can construct a solution based on q-Weyl algebra

Inoue-Kuniba-Sun-Terashima-Yagi (2024)

$$[u_i, w_j] = \hbar \delta_{ij}, \quad [u_i, u_j] = [w_i, w_j] = 0$$

$$q = e^{\hbar} \qquad i, j = 1, 2, 3$$

$$\mathcal{F} = \bigoplus_{n_i \in \mathbb{Z}} \mathbb{C} |n_1, n_2, n_3\rangle, \quad \mathcal{F}^* = \bigoplus_{n_i \in \mathbb{Z}} \mathbb{C} \langle n_1, n_2, n_3|$$

$$e^{u_k}|n\rangle = q^{n_k}|n\rangle \qquad e^{u_k}\langle n| = q^{n_k}|n\rangle \quad \llcorner^{q}$$

$$e^{w_k}|n\rangle = |n+e_k\rangle \qquad e^{w_k}\langle n| = \langle n-e_k| \quad //$$

$$R \in \mathrm{End}(\mathcal{F})$$

$$R = \Psi_q\left(e^{u_1+u_3+w_1-w_2+w_3}\right)^{-1}\Psi_q\left(e^{u_1-u_3+w_1-w_2+w_3}\right)^{-1}$$

$$P\,\Psi_q\left(e^{u_1-u_3+w_1-w_2+w_3}\right)\Psi_q\left(e^{u_1+u_3+w_1-w_2+w_3}\right)$$

$$\Psi_q(Y) = \frac{1}{(-qY;q^2)_\infty} = \sum_{n=0}^{\infty}\frac{(-qY)^n}{(q^2;q^2)_n}$$

quantum dilog. = q-exponential

$$P = e^{\hbar^{-1}(u_3-u_2)w_1}\,\rho_{23}$$ $\leftarrow$ transposition

Prop [IKSTY (2024)]

The matrix elements $R^{n_1 n_2 n_3}_{n_1' n_2' n_3'} = \langle \mathbf{n} | R | \mathbf{n}' \rangle$ are

$$R^{n_1 n_2 n_3}_{n_1' n_2' n_3'} = \delta^{n_1+n_2}_{n_1'+n_2'} \delta^{n_2+n_3}_{n_2'+n_3'} \frac{q^{n_1 n_3 + n_2'}}{2\pi i}$$

$$\oint \frac{dz}{z^{n_2'+1}} \frac{(-z q^{2+n_1'+n_3'} ; q^2)_\infty (-z q^{-n_1-n_3} ; q^2)_\infty}{(-z q^{n_1'-n_3'} ; q^2)_\infty (-z q^{n_3'-n_1'} ; q^2)_\infty}$$

$$= \delta^{n_1+n_2}_{n_1'+n_2'} \delta^{n_2+n_3}_{n_2'+n_3'} (-1)^{n_2'} q^{n_1 n_3 + n_2 (n_3' - n_1')}$$

$$\binom{n_1'+n_2'}{n_1'}_{q^2} \, {}_2\phi_1 \left(\begin{matrix} q^{-2n_2'}, \, q^{-2n_1} \\ q^{-2(n_1'+n_2')} \end{matrix} ; q^2, q^{-2n_3'} \right)$$

§4 Relation to supersymmetric gauge theory

Awata - Hasegawa - K - Ohkawa - Shakirov - Shiraishi - Yamada

(2023 + work in progres)

Non stationary difference equation

for the instanton counting partition function

on $\mathbb{R}^4 \times S^1_q$ $\quad q \sim e^{\hbar R}$ $\quad R$: radius of S^1

$$\Psi(\Lambda, x) = \sum_{k,l \geq 0} c_{k,l} \, x^k \left(\frac{\Lambda}{x}\right)^l$$

$$c_{k,l} = c_{k,l}(Q, d_1, \ldots d_4 ; q, t)$$

$$T_{qtQ,x}\, T_{t,\Lambda}\, \Psi(\Lambda, x) = \mathcal{H}_S(d_1, \ldots d_4; q)\, \Psi(\Lambda, x)$$

$$T_{t,\Lambda} : \Lambda \longrightarrow t\Lambda \qquad \text{"time" shift}$$

When $d_2 = q^{-n}$, $d_4 = q^{-m}$ $\quad n, m \in \mathbb{Z}_{\geq 0}$

$$\Psi(\Lambda, x) \Rightarrow \sum_{i=-n}^{m} \tilde{c}_i(\Lambda)\, x^i$$

consistent truncation to a finite series in x.

$$\mathcal{H}_S\, x^i = \sum_{j=-n}^{m} \underline{r_{i,j}(\Lambda, d_1, d_3; q)}\, x^j$$

$\uparrow$

R-matrix of $U_q(\widehat{sl_2})$

with spin $N = n+m$

<u>Observation</u> $r_{i,j}(\Lambda; d_1, d_3 : q)$ is given by

the trace of the product of 3D R-matrix

(up to a similarity transformation or a base change.)

$$B_N = \{ \mathbb{i} = (i_1, i_2) \in \mathbb{Z}_{\geq 0} \ ; \ i_1 + i_2 = N \}$$

$$|B_N| = N+1$$

R^{abc}_{ijk} : matrix elements of 3D R matrix

$$R^{tr_3}(z)^{\mathbb{a}, \mathbb{b}}_{\mathbb{i}, \mathbb{j}} = \sum_{c_1, c_2} z^{c_1} R^{a_1 b_1 c_1}_{i_1 j_1 c_2} R^{a_2 b_2 c_2}_{i_2 j_2 c_1}$$

$$\mathbb{a}, \mathbb{b}, \mathbb{i}, \mathbb{j} \in B_N$$

By the conservation law $\mathbf{i} + \mathbf{j} = \mathbf{a} + \mathbf{b} \Rightarrow \mu$

$$\left(R^{tr_3}(z)^{\mathbf{a}\,\mathbf{b}}_{\mathbf{i}\,\mathbf{j}} \sim \delta^{a_1+b_1}_{i_1+i_2} \, \delta^{a_2+b_2}_{i_2+j_2} \right)$$

$$R^{tr_3}(z)^{\mathbf{a}\,\mathbf{b}}_{\mathbf{i}\,\mathbf{j}} = R^{tr_3}(z;\mu)^{\mathbf{a}}_{\mathbf{i}} \qquad (N+1) \times (N+1) \text{ matrix.}$$

$$\sim r_{i,j}(\Lambda, d_1, d_3 ; q)$$

$$\left(\Lambda \Rightarrow z \qquad d_1 = q^{-\mu_1}, \; d_3 = q^{-\mu_2} \right)$$

A formal analytic continuation to complex parameters $d_1 \dots d_4$ //

Continuous or unbounded spin variables imply

a third direction of space ??