Exact renormalization flow for matrix product density operators

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Appearing on arXiv soon (hopefully…)

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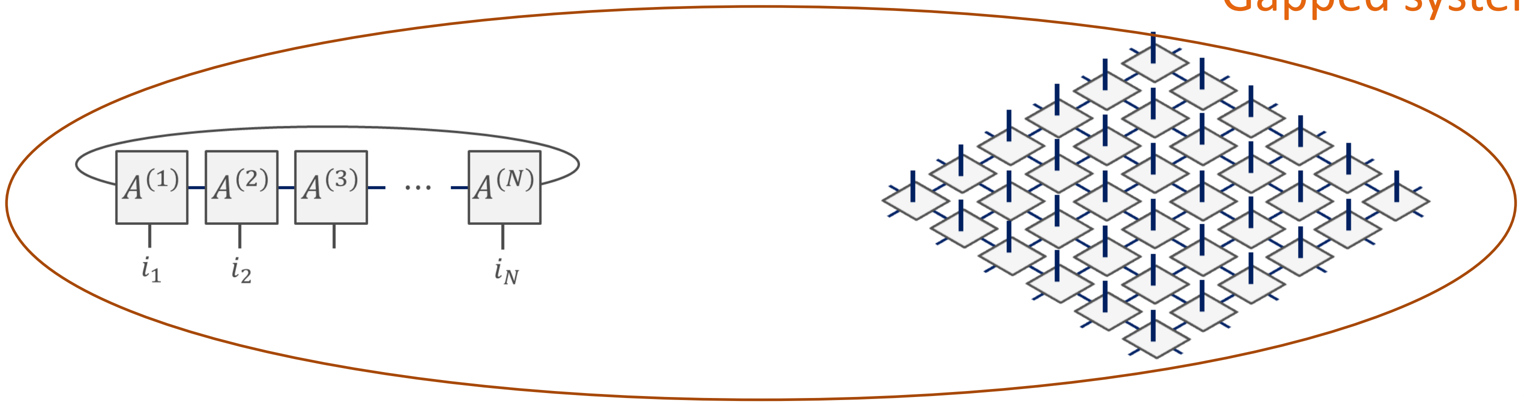
Tensor Networks

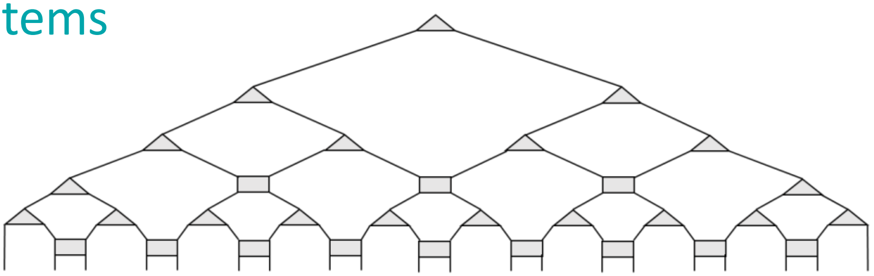
$$|\psi\rangle = \sum_{i_1,..,i_N} c_{i_1 i_2 \ldots i_N} \, |i_1 \ldots i_N\rangle$$

Gapped systems

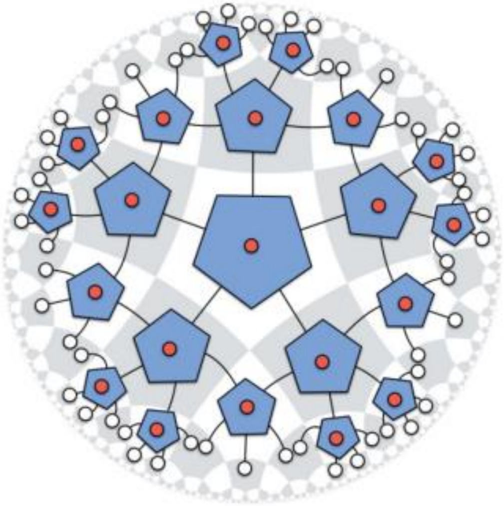
$$c_{i_1 i_2 \ldots i_N} =$$



Critical systems



High-energy physics

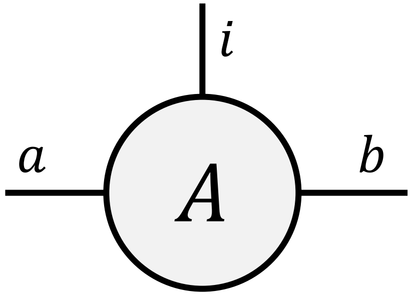


Various applications in quantum physics

[from arXiv:1802.01040]

Contraction rule

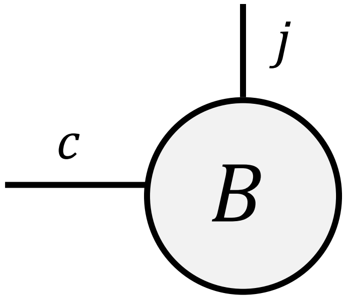
- Open leg = index of the tensor



$$A^i_{ab} \in \mathbb{C}$$

$$i = 1, \dots, d$$
$$a, b = 1, \dots, D$$

$A^i \coloneqq \left(A^i_{ab}\right)$: matrix

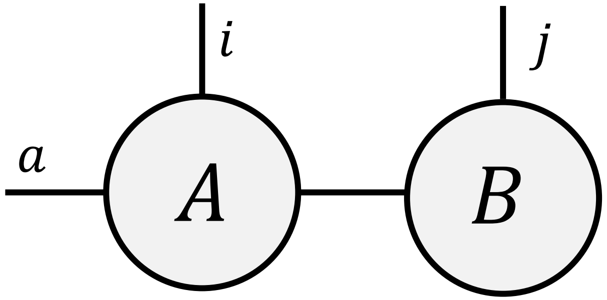


$$B^j_c \in \mathbb{C}$$

$$i = 1, \dots, d'$$
$$c = 1, \dots, D$$

$|B^j\rangle \coloneqq \left(B^j_c\right)$: vector

- Connected leg = sum over the index



$$\sum_{b=1}^{D} A^i_{ab} B^j_b \in \mathbb{C},$$

vector

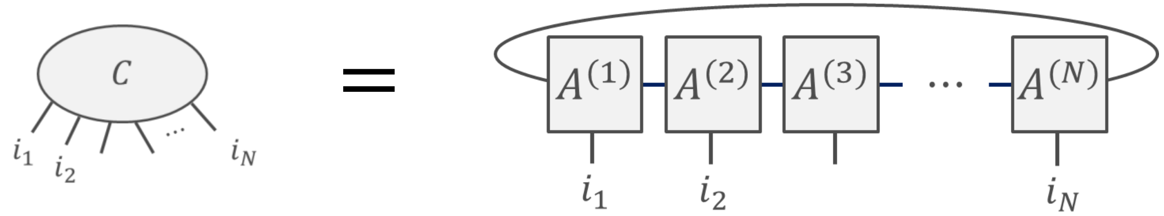
$$A^i |B^j\rangle = \left(\sum_{b=1}^{D} A^i_{ab} B^j_b \right)$$

Matrix Product states

$$|\psi\rangle = \sum_{i_1,\ldots,i_N} c_{i_1\ldots i_N} |i_1 i_2 \ldots i_N\rangle \in \mathbb{C}^{d^{\otimes N}}$$

MPS $$|\psi\rangle = \sum_{i_{1,\ldots,}i_N} \mathrm{Tr}\left(A_{i_1}^{(1)} A_{i_2}^{(2)} \ldots A_{i_N}^{(3)}\right) |i_1 \ldots i_N\rangle$$

$A_{i_k}^{(j)}$: $D \times D$ matrix (for each i_k, j)



- The number of parameters needed to specify a MPS $= dND^2 \ll d^N$
- Always satisfies an area law of entanglement: $S(X)_\psi \coloneqq -\mathrm{Tr}\rho_X \log\rho_X \leq \log D$

What are these states?

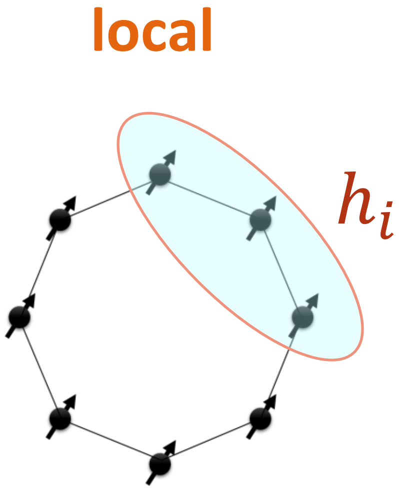
MPS and 1D gapped physics

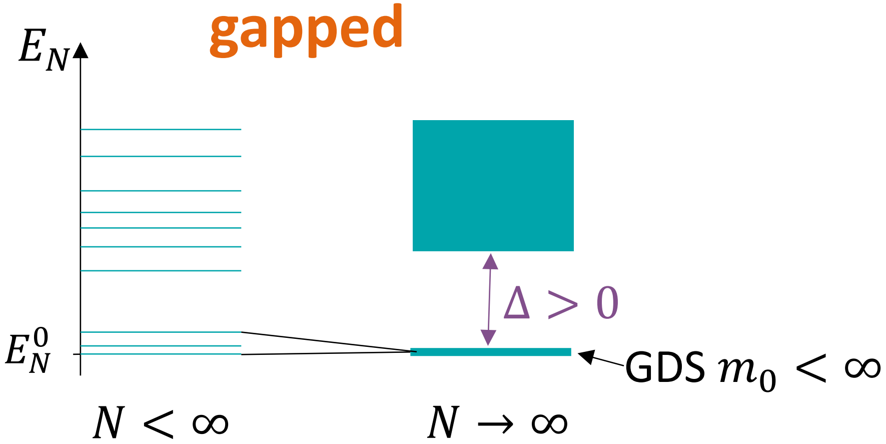
- ▶ **1D frustration-free, local, gapped Hamiltonian:**

$$H = \sum_i h_i$$

frustration-free

$$h_i|\psi_{GS}\rangle = 0, \forall i.$$





MPS ⊃ 1D local gapped frustration-free ground states

- Any 1D gapped ground state can be approximated by a MPS [Hastings, '07; Arad *et al.*, '13]

$$D \sim \mathrm{poly}(N, 1/\epsilon)$$

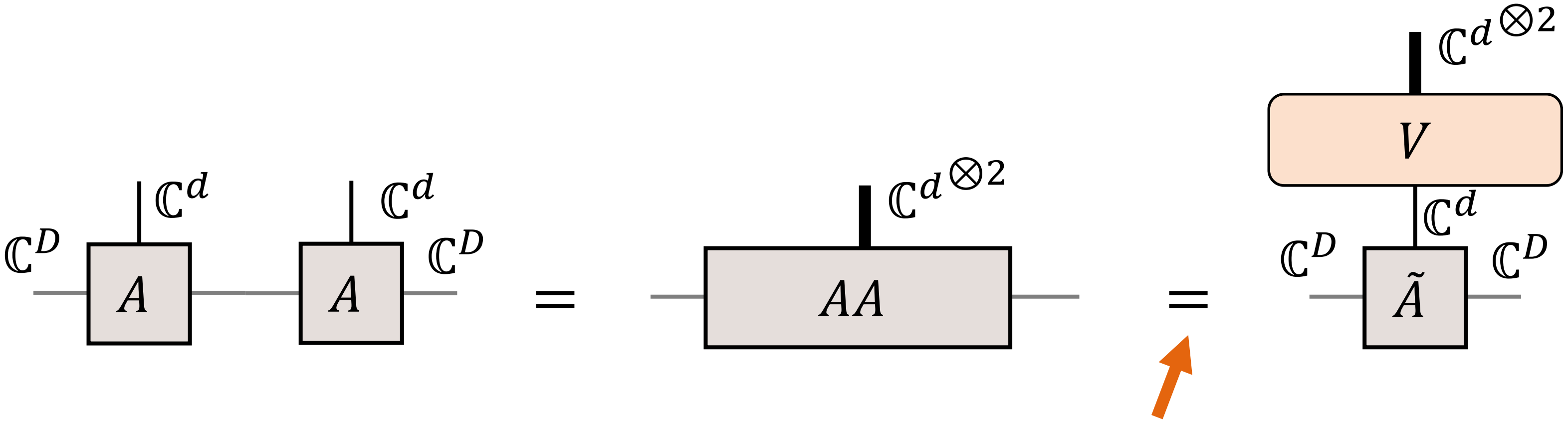
MPS ⊂ 1D local gapped frustration-free ground states

- Any MPS has a 1D local, gapped Hamiltonian H s.t. the MPS is a ground state of H

[Fannes, *et al.*, '92; Nachtergaele, '96]

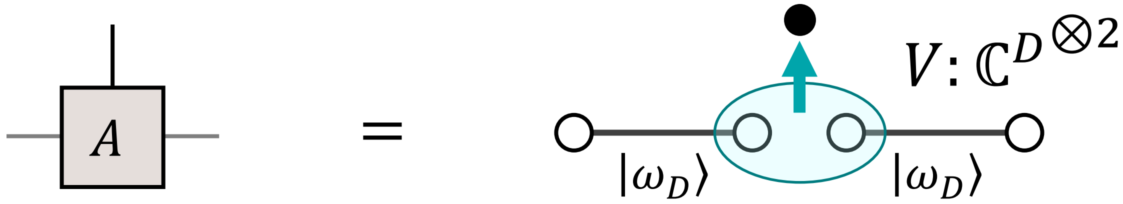
Renormalization Group flow of MPS

- ▶ MPS has a physically reversible coarse-graining operation.



Use the polar decomposition of tensor $AA = V\tilde{A}\ (d > D^2\ w.l.o.g.)$.

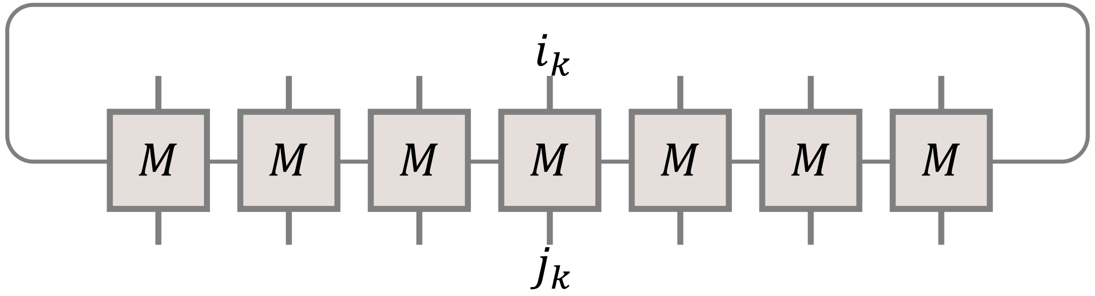
- ▶ The RG-fixed point is achieved by iteration → Isometric MPS



$V: \mathbb{C}^{D\otimes 2} \to \mathbb{C}^d$ isometry

- ▶ The RG-fixed point is useful to characterize quantum gapped phases [Schuch et al., '11]

Matrix Product Density Operators (MPDO)



$\rho_{MPDO} = \sum_{\boldsymbol{i},\boldsymbol{j}} \mathrm{Tr}\left(M^{i_1 j_1} M^{i_2 j_2} \dots M^{i_N j_N}\right) |i_1 i_2 \dots i_N\rangle\langle j_1 j_2 \dots j_N|$ $M_{i_k j_k}: D \times D$ matrix (for each i_k, j_k)

- A natural generalization of Matrix Product States to 1D **mixed states**.
- A good ansatz for thermal states and steady states in 1D systems.

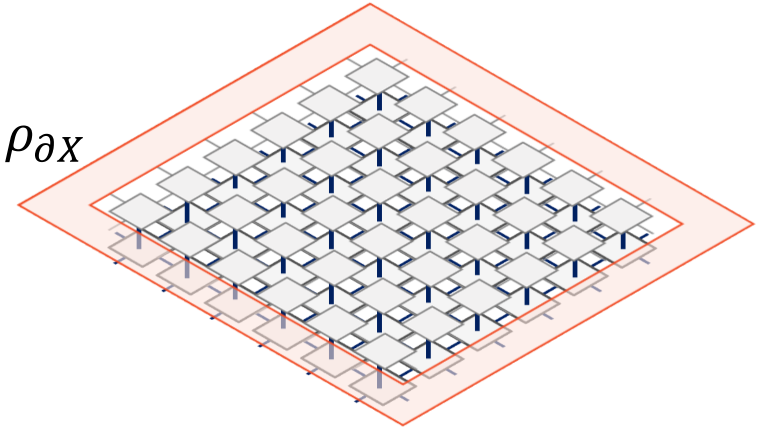
▶ **Any Gibbs states of 1D local Hamiltonian can be approximated by a MPDO [Hastings '06].**

$$\text{MPDO} \supseteq \rho_{Gibbs} = \frac{1}{Z} e^{-\beta \sum_i h_{i,i+1}}$$

MPDOs ≠ Gibbs states

- **MPDO can describe more than just Gibbs states.**

PEPS
(2D pure states)



D-dimensional *pure* states ⟷ $(D-1)$-dimensional *mixed* states

- **Boundary states of 2D topological order can be non-thermal MPDOs.**

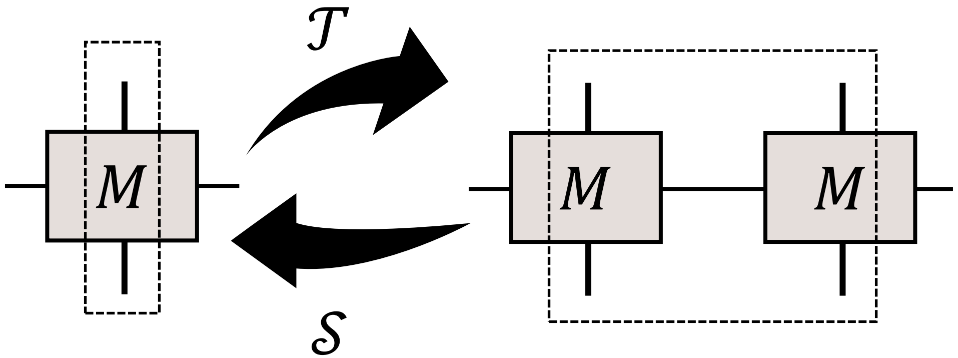
Boundary of toric code model

$$\rho_{\mathrm{MPDO}} = \frac{1}{2^n}\left(I^{\otimes n} + Z^{\otimes n}\right) \neq \frac{1}{Z} e^{-\beta \sum_i h_{i,i+1}}$$

Renormalization fixed-points of MPDO

- A MPDO is called a fixed-point MPDO if there is a pair of **CPTP-maps** $\mathcal{S}, \mathcal{T}$ such that

$$M = \sum_{ij} |i\rangle\langle j| \otimes M^{ij}$$



$$\sum_{ij} \mathcal{S}(|i_1 i_2\rangle\langle j_1 j_2|) \otimes M^{i_1 j_1} M^{i_2 j_2} = M, \qquad \sum_{ij} \mathcal{T}(|i\rangle\langle j|) \otimes M^{ij} = \sum_{ij} |i_1 i_2\rangle\langle j_1 j_2| \otimes M^{i_1 j_1} M^{i_2 j_2}$$

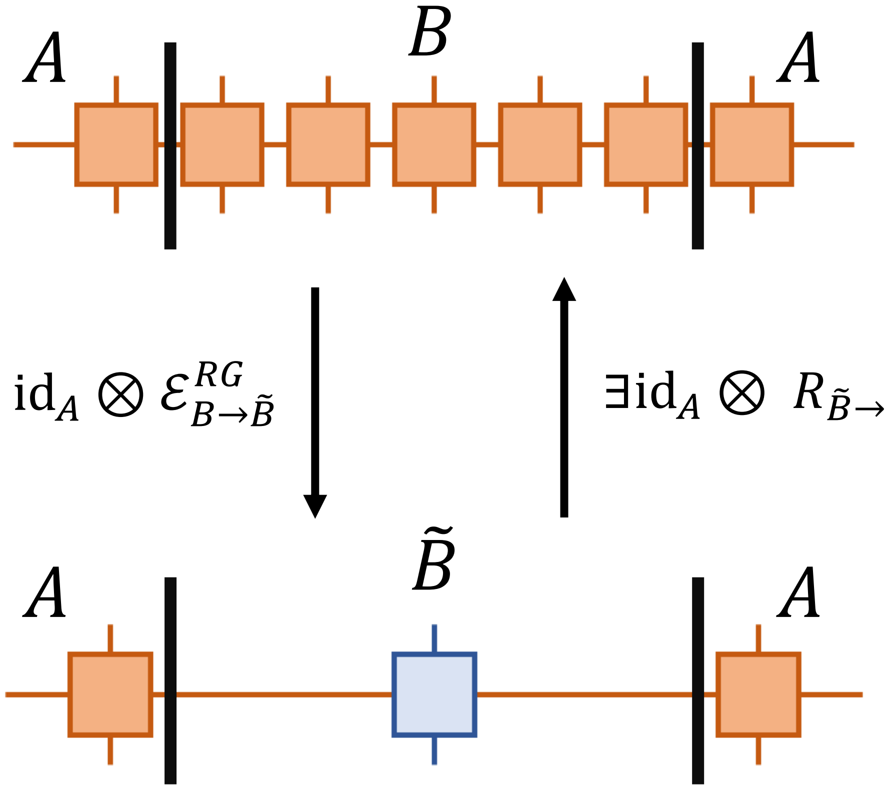
Theorem [Cirac, et al., ‘17] **:**
If ρ is a **fixed-point MPDO**, then ρ is **a “global MPO”** × **a commuting Gibbs state.**

Matrix Product Operator

$$\rho_{\mathrm{MPDO}} = \bigoplus_{i=1}^{d} \lambda_i P_i \, e^{-\beta \sum_k h_{k,k+1}} \qquad \left[P_i, \sum_k h_{k,k+1}\right] = [h_{k,k+1}, h_{l,l+1}] = 0.$$

Caveat: A notion of **renormalization flow is missing** for these “fixed-points”.

Exact (reversible) RG-flow of MPDO?



CPTP-map $\mathrm{id}_A \otimes \mathcal{E}^{RG}_{B\to\tilde{B}}$

$\exists \mathrm{id}_A \otimes R_{\tilde{B}\to B}$

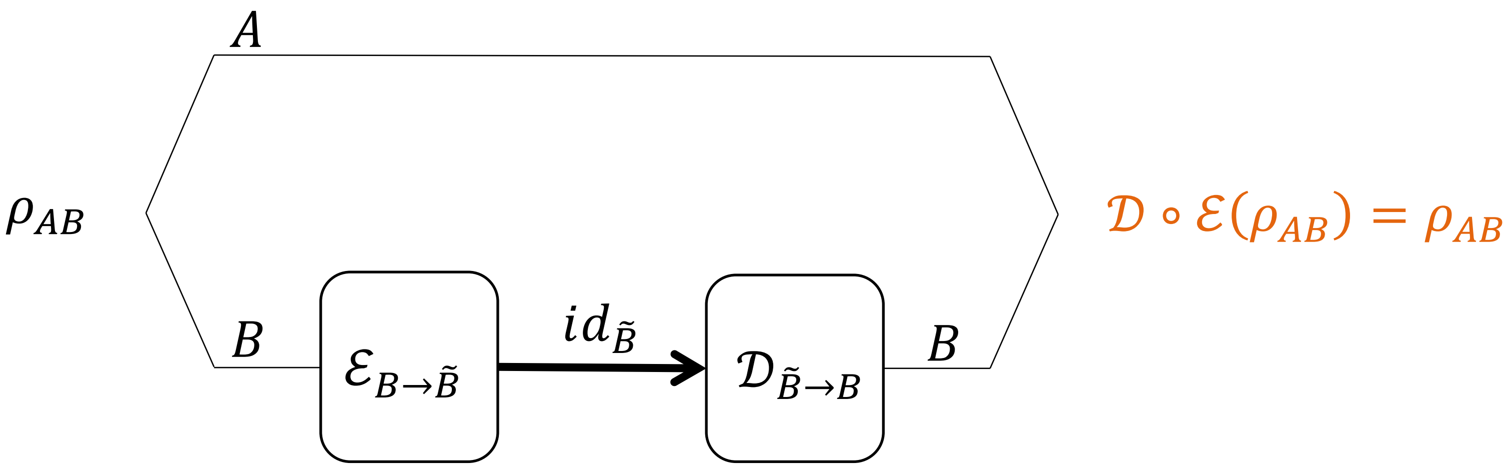
for arbitrary large A

$|\tilde{B}| < |B|$

If $|\tilde{B}|$ can chosen to be independent of $|B|$, then we obtain the desired RG-flow.

Exact compression of general bipartite states

- **One-shot, exact** compression of **mixed bipartite state**



Question

What is the minimum dimension of $\mathcal{H}_{\tilde{B}}$?

Minimal sufficient subalgebra

The condition $\mathcal{D}_{\tilde{B}\to B}\circ\mathcal{E}_{B\to\tilde{B}}(\rho_{AB})=\rho_{AB}$ is equivalent to the following **[Hayden, et al., ‘04]:**

$$\mathcal{D}_{\tilde{B}\to B}\circ\mathcal{E}_{B\to\tilde{B}}(\mu_B)=\mu_B,\forall\mu_B\in\mathcal{S},\qquad \mathcal{S}:=\left\{\mu_B=\frac{\mathrm{tr}_A(O_A\rho_{AB})}{\mathrm{tr}(O_A\rho_A)}\,\middle|\,0\le O_A\le I_A\right\}.$$

- The minimal dimension of $\widetilde{B}$ is then derived from the **minimum sufficient subalgebra** of $\mathcal{S}$. **[Petz, ‘86, ‘88][Jenčová&Petz, ‘06]**

$$\mathcal{M}_B^S:=\mathrm{Alg}\{\mu_B^{it}\rho_B^{-it},\mu\in\mathcal{S},t\in\mathbb{R}\}\subset\mathcal{B}(\mathcal{H}_B)$$

This is a finite-dimensional C^*-algebra, thus there is a decomposition

$$\mathcal{H}_B\cong\bigoplus_i\mathcal{H}_{B_i^L}\otimes\mathcal{H}_{B_i^R}\qquad\text{s.t.}\qquad\mathcal{M}_B^S\cong\bigoplus_i\mathrm{Mat}\left(\mathcal{H}_{B_i^L},\mathbb{C}\right)\otimes I_{B_i^R}.$$

Minimal sufficient subalgebra (cont.)

- For any bipartite state $\rho_{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, s.t., $\rho_B > 0$,

$$\mathcal{H}_B \cong \bigoplus_i \mathcal{H}_{B_i^L} \otimes \mathcal{H}_{B_i^R} \qquad \text{s.t.} \qquad \mathcal{M}_B^S \cong \bigoplus_i \mathrm{Mat}\left(\mathcal{H}_{B_i^L}, \mathbb{C}\right) \otimes I_{B_i^R}$$

and

$$\rho_{AB} = \bigoplus_i p_i \underbrace{\overbrace{\rho_{AB_i^L}}^{\text{Quantumly correlated}} \otimes \omega_{B_i^R}}_{\text{Classically correlated}} .$$

Sometimes called “Koashi-Imoto decomposition”. **[Koashi, Imoto, ‘02][Hayden, et al., ‘04]**

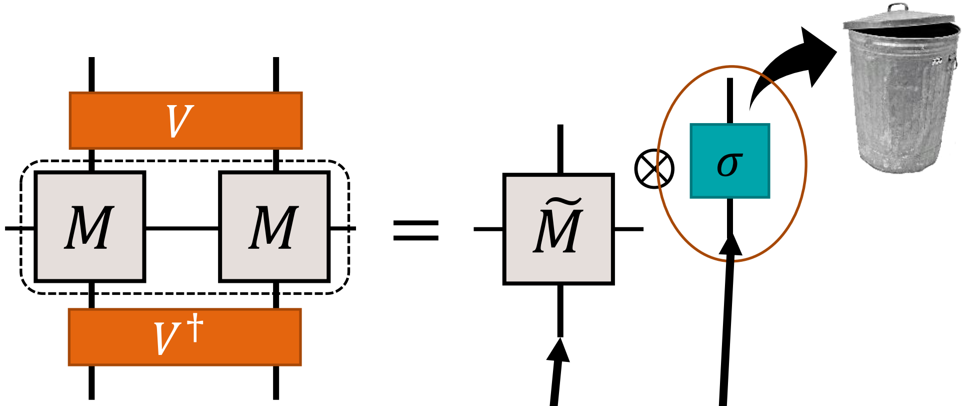
- The minimal exact compression is then given by

$$\mathcal{E}_{B\to\tilde{B}}: \rho_{AB} \mapsto \rho_{A\tilde{B}} := \bigoplus_i p_i \, \rho_{AB_i^L} .$$

Exact (reversible) RG-flow of MPDO?

- **We need to establish RG-flow (coarse-graining) for MPDOs**

Unlike RG-flow for MPS (which is well-defined), one needs to reduce the entropy to keep the local dimension to be a constant.



This is exactly given by **the Koashi-Imoto decomposition!**

$$\rho_{RA} = \bigoplus_i p_i\, \rho_{RA_i^L} \otimes \omega_{A_i^R}$$

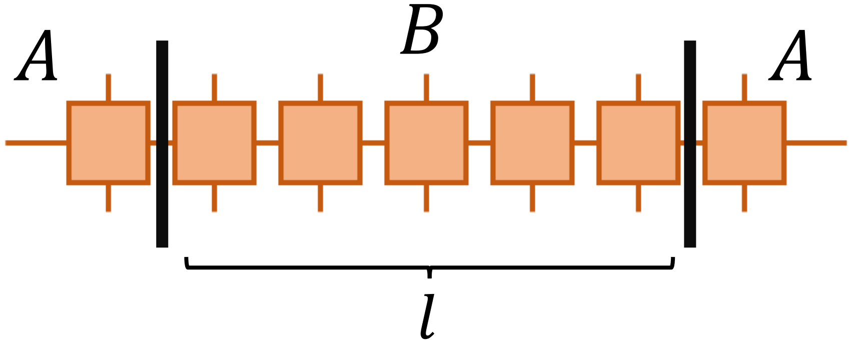
Diverging RG-flow

- **We show not all MPDOs admit RG-flow. Consider an MPDO**

$$\rho^{(L)} := \frac{1}{3^L}\left(I^{\otimes L} + \Lambda^{\otimes L}\right)$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$\mathrm{tr}\Lambda = 0, \|\Lambda\| \leq 1,$
$\lambda_1 \neq \lambda_2 \neq \lambda_3\,.$



The minimal sufficient subalgebra for $\mathcal{S} := \left\{\mu_B = \frac{\mathrm{tr}_A(O_A\rho_{AB})}{\mathrm{tr}(O_A\rho_A)} \middle| 0 \leq O_A \leq I_A\right\}$ is

$$\mathcal{M}_B^S = \mathrm{Alg}\{I^{\otimes l}, \Lambda^{\otimes l}\} \cong \mathbb{C}^{\mathrm{poly}(l)}$$

Exact RG-flow must diverge!

MPDO with a RG-flow

▶**We thus consider a subclass of MPDOs with a RG-flow.**

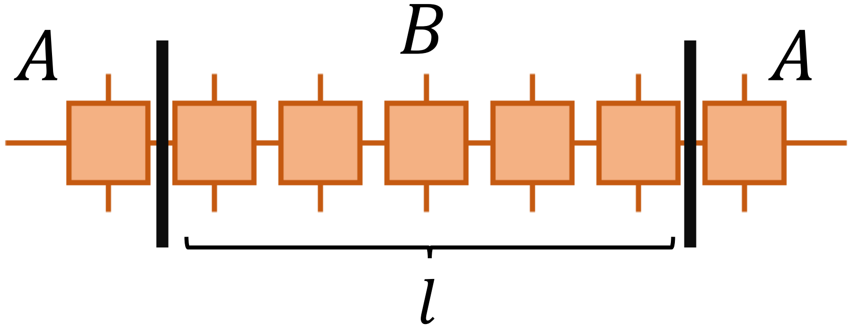
Condition 1: there is a finite-dimensional C^*-algebra $\mathcal{A}$

$$\mathcal{A} = \bigoplus_a \mathrm{Mat}_{d_a}(\mathbb{C})$$

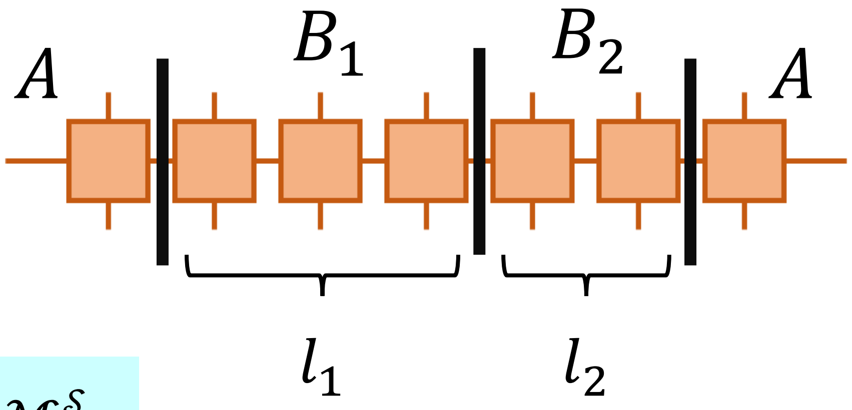
and injective representations $\{\pi_l\}$ s.t.

l - independent constant dimension

$$\mathcal{M}_B^{\mathcal{S}} = \pi_l(\mathcal{A}) \cong \bigoplus_a \mathrm{Mat}_{d_a}(\mathbb{C}) \otimes I_{d_a^{(l)}} \qquad \forall B, |B| = l.$$



The inclusion relation



Lemma: $\mathcal{M}^{\mathcal{S}}_{B_1B_2} \subset \mathcal{M}^{\mathcal{S}}_{B_1} \otimes \mathcal{M}^{\mathcal{S}}_{B_2}$.

The inclusion $\iota_{l_1+l_2}\colon \mathcal{M}^{\mathcal{S}}_{B_1B_2} \hookrightarrow \mathcal{M}^{\mathcal{S}}_{B_1} \otimes \mathcal{M}^{\mathcal{S}}_{B_2}$ induces

$$\Delta_{l_1+l_2} \coloneqq \left(\pi_{l_1}^{-1} \otimes \pi_{l_2}^{-1}\right) \circ \iota_{l_1+l_2} \circ \pi_{l_1+l_2}$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, red, "\Delta_{l_1+l_2}"'] \arrow[d, "\pi_{l_1+l_2}"'] & \mathcal{A} \otimes \mathcal{A} \\
\mathcal{M}^{\mathcal{S}}_{B_1B_2} \arrow[r, hook, "\iota_{l_1+l_2}"] & \mathcal{M}^{\mathcal{S}}_{B_1} \otimes \mathcal{M}^{\mathcal{S}}_{B_2} \arrow[u, "\pi_{l_1}^{-1} \otimes \pi_{l_2}^{-1}"']
\end{tikzcd}$$

MPDO with a RG-flow (definition)

▶**We say a MPDO has a $(\boldsymbol{\pi_l}, \boldsymbol{\mathcal{A}})$ RG-flow if it satisfies the following two conditions.**

Condition 1: $\mathcal{M}_B^{\mathcal{S}} = \pi_l(\mathcal{A}), \ \ \forall B, |B| = l.$

Condition 2: $\exists \Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ **s. t.** $\Delta_{\mathrm{l}_1+\mathrm{l}_2} = \Delta, \qquad \forall l_1, l_2 \in \mathbb{N}.$

Proposition:

The linear map $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ becomes a ***comultiplication,*** i.e., it satisfies

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta =: \Delta^2.$$

Pre-bialgebra behind RG-flows

In addition to comultiplication $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, we show that $\mathcal{A}$ has a **counit $\boldsymbol{\epsilon}$:**

$$\epsilon: \mathcal{A} \to \mathbb{C}, \text{ s.t. } (\mathrm{id} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}.$$

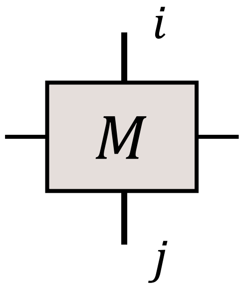
Theorem:

The algebra $\mathcal{A}$ associated to $(\pi_l, \mathcal{A})$ is a **pre-bialgebra.**

pre-bialgebra = algebra ∧ co-algebra with multiplicative coproduct: $\Delta(xy) = \Delta(x)\Delta(y)$.

Sketch of the proof:

counit of $\mathcal{A}$ ⇔ unit of $\mathcal{A}^*$, the dual space (which becomes algebra by Δ)



$$M = \sum_{ij} |i\rangle\langle j| \otimes \overset{\psi(\mathcal{A}^*)}{\overset{\rotatebox{90}{$\in$}}{M^{ij}}} = \sum_{\alpha,\beta} \overset{\pi(\mathcal{A})}{\overset{\rotatebox{90}{$\in$}}{W^{\alpha\beta}}} \otimes |\alpha\rangle\langle\beta|$$

ψ: injective rep.

M^{ij} generates a unital algebra (by a property of tensor network) → $\mathcal{A}^*$ must contain a unit.

Structure theorem

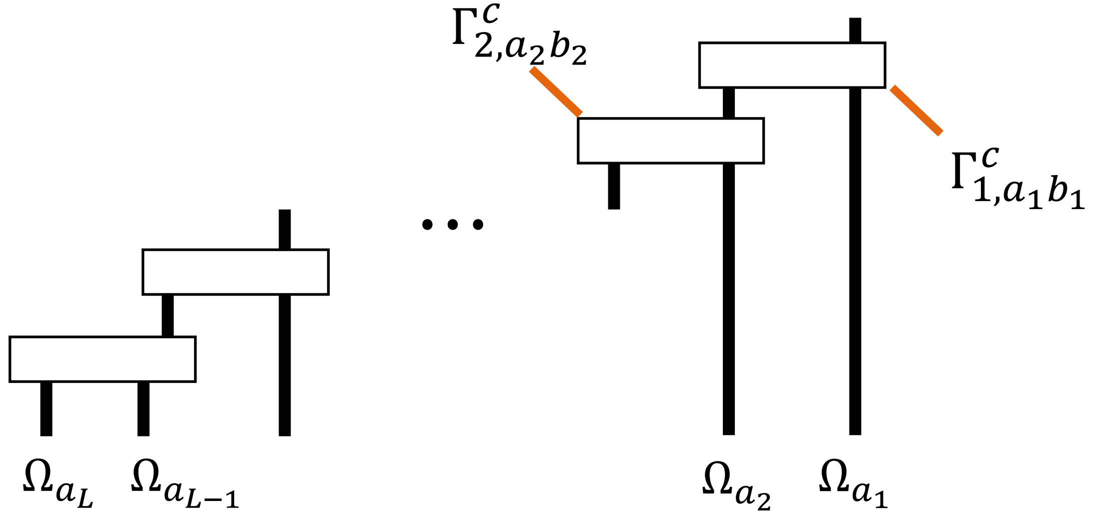
Theorem: Any MPDO $\rho \in \mathcal{B}\left(\left(\mathbb{C}^d\right)^{\otimes L}\right)$ with a $(\pi_l, \mathcal{A})$ RG-flow can be written as

$$\rho = \pi^{\otimes L} \circ \Delta^{L-1}\left(w^{(L)}\right)\Omega^{(L)}, \qquad \exists w^{(L)} \in \mathcal{A},$$

where $\left[\pi^{\otimes L} \circ \Delta^{L-1}(a), \Omega^{(L)}\right] = 0, \forall a \in \mathcal{A}.$

$$\Omega^{(L)} = \bigoplus_{c,\boldsymbol{a},\boldsymbol{b}} \Gamma^{c}_{1,a_1,b_1} \otimes \Gamma^{b_1}_{2,a_2,b_2} \cdots \otimes \Gamma^{b_{L-2}}_{L-1,a_{L-1},a_L} \otimes_k^L \Omega_{a_k} \in \mathcal{B}\left(\mathbb{C}^{d^L}\right)$$

$$\Gamma^{c}_{l,a,b}, \Omega_{a_i} \geq 0$$



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where $\left[\pi^{\otimes L} \circ \Delta^{L-1}(a), \Omega^{(L)}\right] = 0, \forall a \in \mathcal{A}.$

Recall that the structure theorem on the fixed-point is given as

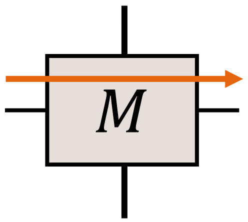
$$\rho_{\text{fixedpoint}} = \bigoplus_{i=1}^{d} \lambda_i P_i e^{-\beta \sum_k h_{k,k+1}} \qquad \left[P_i, \sum_k h_{k,k+1}\right] = \left[h_{k,k+1}, h_{l,l+1}\right] = 0.$$

Proof: KI decomposition and canonical form

- Each tensor has a canonical block form (up to a gauge transformation).

$$M^{ij} \mapsto X M^{ij} X^{-1}$$

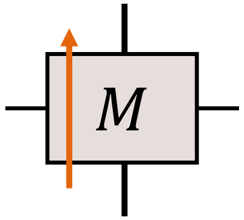
Horizontal canonical form



$$M = \sum_{ij} |i\rangle\langle j| \otimes M^{ij}$$

$$M^{ij} = \bigoplus_k \mu_k M_k^{ij} \cong \bigoplus_a M_a^{ij} \otimes N_a \qquad (N_a)_{\eta\eta'} := \delta_{\eta\eta'}\mu_\eta$$

Proposition [Cirac et al., '17]: M is also in a canonical form in vertical direction



$$M = \sum_{\alpha\beta} W^{\alpha\beta} \otimes |\alpha\rangle\langle\beta|$$

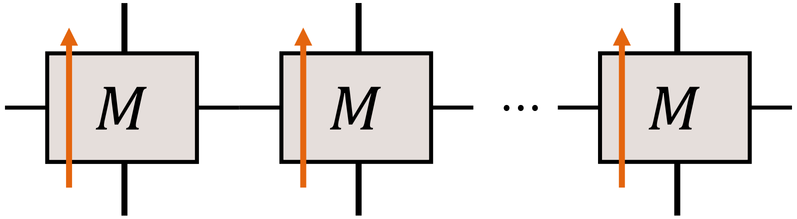
$$W^{\alpha\beta} \cong \bigoplus_a W_a^{\alpha\beta} \otimes \Omega_a$$

KI-decomposition of MPDO

$$\rho_{AB} = \bigoplus_i p_i\, \rho_{AB_i^L} \otimes \omega_{B_i^R}\,.$$

Proof: KI decomposition and canonical form

Condition 1: $\mathcal{M}_B^{\mathcal{S}} = \pi_l(\mathcal{A}), \ \forall B, |B| = l.$



$$MM \dots M =: \sum_{\alpha\beta} W^{\alpha\beta}[l] \otimes |\alpha\rangle\langle\beta|$$

$$W^{\alpha\beta}[l] \cong \bigoplus_a \pi_l\left(\widehat{w}_a^{\alpha\beta}[l]\right) \otimes \Omega_a^{(l)}.$$

By definition and **Condition 2**,

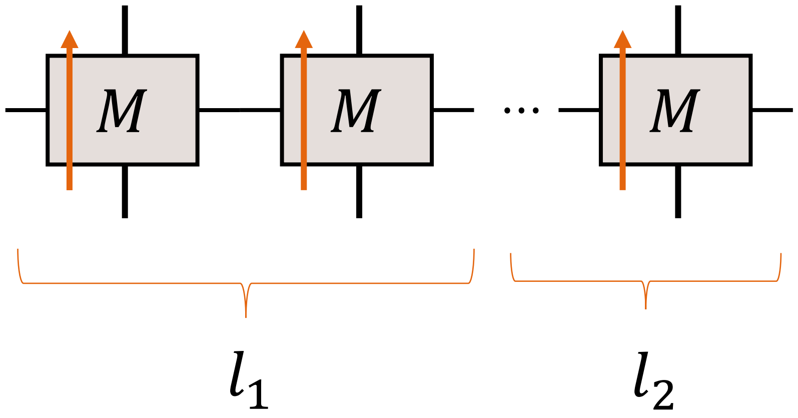
Omit the inclusion maps.

$$\iota_{l_1+l_2} \circ \pi_{l_1+l_2} = \left(\pi_{l_1} \otimes \pi_{l_2}\right) \circ \Delta \quad \Rightarrow \pi_l = \pi_1^{\otimes l} \circ \Delta^{l-1}.$$

$$W^{\alpha\beta}[l] \cong \bigoplus_a \pi_1^{\otimes l} \circ \Delta^{l-1}\left(\widehat{w}_a^{\alpha\beta}[l]\right) \otimes \Omega_a^{(l)}.$$

Proof: KI decomposition and canonical form

$$W^{\alpha\beta}[l] \cong \bigoplus_a \pi_1^{\otimes l} \circ \Delta^{l-1}\left(\widehat{w}_a^{\alpha\beta}[l]\right) \otimes \Omega_a^{(l)}.$$



$$W^{\alpha\beta}[l_1 + l_2] = \sum_\gamma W^{\alpha\gamma}[l_1] \otimes W^{\gamma\beta}\,[l_2]$$

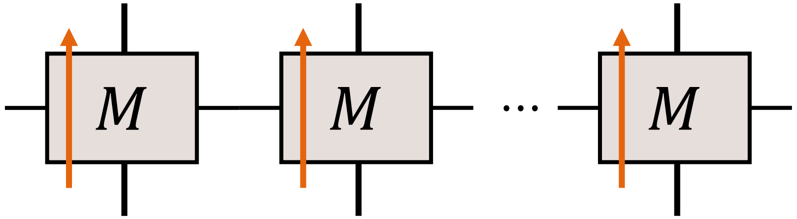
Consistency between decomposition (*) for LHS and RHS

Lemma: $\sum_\gamma \mathrm{V}_{\mathrm{ab}} \left(\pi_{l_1}\left(\widehat{w}_a^{\alpha\gamma}[l_1]\right) \otimes \pi_{l_2}\left(\widehat{w}_b^{\gamma\beta}[l_2]\right)\right) V_{ab}^\dagger = \oplus_c \Gamma_{ab}^c\,[l_1, l_2] \otimes \Omega_l^c$

This lemma factorizes $\Omega_a^{(l)}$ into small pieces.

$$\Omega^{(L)} = \bigoplus_{c,\boldsymbol{a},\boldsymbol{b}} \Gamma^c_{1,a_1,b_1} \otimes \Gamma^{b_1}_{2,a_2,b_2} \cdots \otimes \Gamma^{b_{L-2}}_{L-1,a_{L-1},a_L} \bigotimes_k^L \Omega_{a_k} \in \mathcal{B}\left(\mathbb{C}^{d^L}\right)$$

Construction of exact RG-transformation



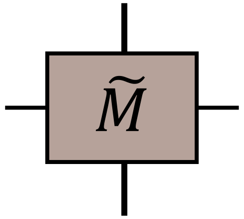
$$W^{\alpha\beta}[l] \cong \bigoplus_{a} \pi_1^{\otimes l} \circ \Delta^{l-1}\left(\widehat{w}_a^{\alpha\beta}[l]\right) \otimes \Omega_a^{(l)}.$$

l - independent constant dimension

$$MM \ldots M =: \sum_{\alpha\beta} W^{\alpha\beta}[l] \otimes |\alpha\rangle\langle\beta|$$

$$\mathcal{E}_{l\to 1} \Big\downarrow \quad \Big\uparrow \mathcal{F}_{1\to l}$$

$\mathcal{E}_{l\to 1}$, $\mathcal{F}_{1\to l}$:CPTP-maps



$$\widetilde{M} =: \sum_{\alpha\beta} \widetilde{W}^{\alpha\beta} \otimes |\alpha\rangle\langle\beta|$$

$$\widetilde{W}^{\alpha\beta} \cong \bigoplus_{a} \frac{\mathrm{tr}\Omega_a^{(l)}}{\mathrm{tr}\Omega_a^{(1)}} \pi_1^{new}\left(\widetilde{w}_a^{\alpha\beta}\right) \otimes \Omega_a^{(1)}.$$

$$\pi_1^{new}\left(\widetilde{w}_a^{\alpha\beta}\right) \coloneqq \pi_1^{\otimes l} \circ \Delta^{l-1}\left(\widehat{w}_a^{\alpha\beta}[l]\right)$$

MPO-algebra from $\mathcal{A}$

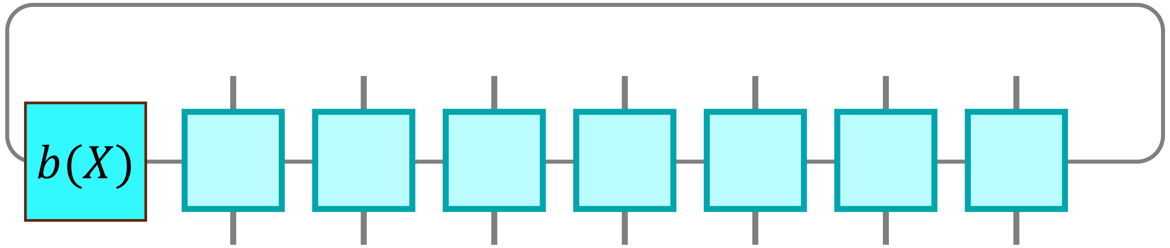
We have seen An exact RG-flow ⇒ a pre-bialgebra $\mathcal{A}$, an injective rep π.

[A. Molnar, et al., ‘22]:

$\pi^{\otimes L} \circ \Delta^{L-1}(\mathcal{A})$ has an MPO-realization of algebra $\mathcal{A}$.

$\forall X \in \mathcal{A}, \exists b(X) \in \mathrm{Mat}_D,$

$$\pi^{\otimes L} \circ \Delta^{L-1}(X) =$$

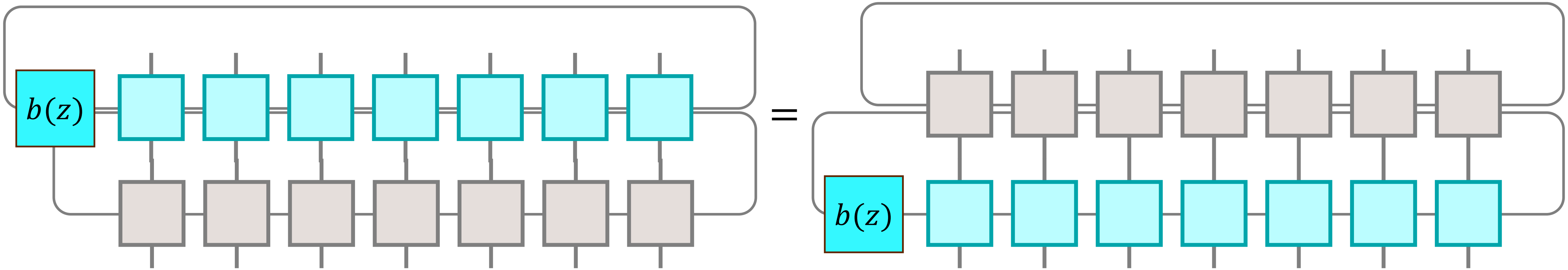


Application: MPO-symmetry

Let $\mathcal{Z}(\mathcal{A})$ be the center of $\mathcal{A}$. For any $z \in \mathcal{Z}(\mathcal{A})$, the MPDO ρ satisfies

$$\left[\pi^{\otimes L} \circ \Delta^{L-1}(z), \rho\right] = \pi^{\otimes L} \circ \Delta^{L-1}\left(\left[z, w^{(L)}\right]\right)\Omega^{(L)} = 0.$$

*Recall that $\left[\pi^{\otimes L} \circ \Delta^{L-1}(a), \Omega^{(L)}\right] = 0, \forall a \in \mathcal{A}$.



Symmetry beyond group representation.

Summary & Discussion

▶ Summary

- We have studied **one-shot** and **exact** data compression of **mixed** quantum source
- We have obtained **a formula for the minimum achievable dimension**

▶ Future direction

Many-boy physics

- Application to **tensor-network states?**

Quantum information

- How about one-shot **approximate** scenario?
- More sophisticated algorithm? Relation to entropic quantities?