On the generic increase of observational entropy in isolated systems

Teruaki Nagasawa, Kohtaro Kato, Eyuri Wakakuwa, Francesco Buscemi

Department of Mathematical Informatics, Nagoya University

September 21, 2024

Introduction

- In his book on the mathematical foundations of quantum theory, John von Neumann introduces and offers an operational motivation for a quantity that is now known as von Neumann (microscopic) entropy.
- However, he notes that this quantity is not the most appropriate to consider in the context of statistical mechanics.
- In order to address this challenge, von Neumann put forth the concept of macroscopic entropy, which considers not only the intrinsic uncertainty inherent to the microscopic state of the system but also the supplementary uncertainty associated with the coarse-grained, macroscopic observation with which the system is being monitored.

Macroscopic states and entropy

Von Neumann derived **von Neumann entropy** (**micro-entropy**).

Definition (von Neumann (micro) entropy)

For quantum state ρ,

$$S(\rho) := -\operatorname{tr}[\rho \log(\rho)]$$

is called **microscopic entropy** or **von Neumann entropy**.

He introduced **macroscopic entropy**, which is different from micro-entropy (This generalisation is observational entropy).

Definition (Observational entropy)

For any POVM $\boldsymbol{P}$ and quantum state ρ,

$$S_{\boldsymbol{P}}(\rho) = -\sum \operatorname{tr}[P_i \rho] \log \frac{\operatorname{tr}[P_i \rho]}{\operatorname{tr}[P_i]}$$

is called **observational entropy**.

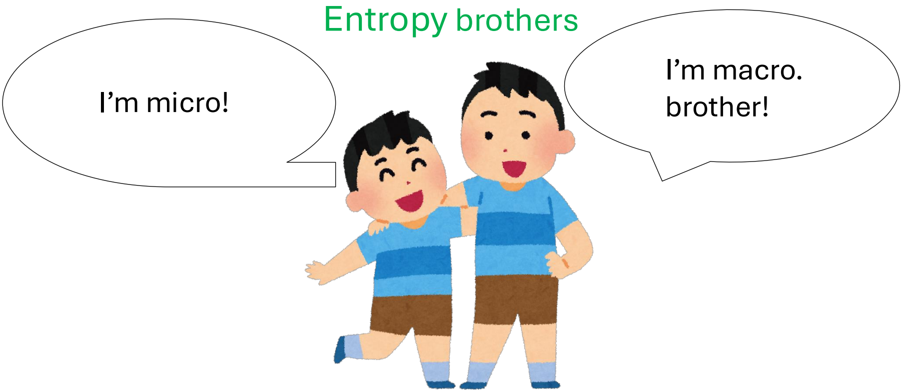

Figure: Von Neumann's microscopic entropy and macroscopic entropy are long-lost brothers.

Macroscopic states and entropy

- Von Neumann's original definition of macroscopic entropy is limited to PVMs.
- In order to show **(quantum) H-theorem**, a proper definition of entropy (i.e., macroscopic entropy) is required.
- Our study aims to recapture these arguments in a modern way (quantum information theory) using observational entropy.

Remark (Classical H-theorem)

Entropy

$$H := -k \int_{\Omega} \rho(X) \log(\rho(X)) dX$$

increases with time evolution (Boltzmann equation), where Ω is a phase space.

Macroscopic states and entropy

Definition (Macroscopic state)

For quantum state ρ and POVM $\boldsymbol{P}$,

$$S_{\boldsymbol{P}}(\rho) = S(\rho)$$

is satisfied, then ρ is called a **macroscopic state**.

The following inequality holds for observational entropy and von Neumann entropy:

$$S_{\boldsymbol{P}}(\rho) \geq S(\rho) \ .$$

For any POVM $\boldsymbol{P}$, macroscopic state ρ, and unitary U,

$$S_{\boldsymbol{P}}(U\rho U^*) \geq S(U\rho U^*) = S(\rho) = S_{\boldsymbol{P}}(\rho) \ .$$

If we take macroscopic state as the initial state, observational entropy will certainly increase with unitary time evolution with probability 1.

When does the OE (*strictly*) increase? In order to answer this question, we first need to provide a characterization of all macroscopic states associated with a given POVM $\boldsymbol{P} = \{P_x\}$.

Theorem

Given a POVM $\boldsymbol{P} = \{P_x\}$, a state $\mathfrak{m}$ is macroscopic for $\boldsymbol{P}$ if and only if there exists a PVM $\boldsymbol{\Pi} = \{\Pi_y\}_y$, with $\boldsymbol{\Pi} \preceq \boldsymbol{P}$, together with coefficients $c_y \geq 0$, such that

$$\mathfrak{m} = \sum_y c_y \Pi_y \ ,$$

where we write $\boldsymbol{Q} \preceq \boldsymbol{P}$ whenever there exists a conditional probability distribution $p(y|x)$ such that $Q_y = \sum_x p(y|x) P_x$, for all y.

Macroscopic states and entropy

- In particular, if $\mathfrak{m}$ is macroscopic for $\boldsymbol{P} = \{P_x\}$, then

$$[\mathfrak{m}, P_x] = 0 \ ,$$

  for all $P_x \in \boldsymbol{P}$.
- This fact demonstrates that a restricted set of unitary operators, those that satisfy the conservation-like relation

$$[U\mathfrak{m}U^\dagger, P_x] = 0 \ , \quad \forall x \ ,$$

  can preserve the observer's information about the system.
- Conversely, a generic evolution, such as one uniformly sampled from the entire set of unitary operators, will necessarily cause a (strict) increase in OE.
- In such cases, although the microscopic evolution is perfectly reversible, from the macroscopic observer's perspective, information is irreversibly lost.

Generic increase of OE

How does observational entropy behave when considering an arbitrary quantum state as the initial state?

Theorem (Haar-random case)

Let us consider a d-dimensional system in an arbitrary (but fixed) state ρ, a POVM $\boldsymbol{P} = \{P_x\}$ with a finite number of outcomes, and a value $\delta > 0$. For a unitary operator U sampled at random according to the Haar (unitarily invariant) distribution, the probability that the system's observational entropy $S_{\boldsymbol{P}}(U\rho U^\dagger)$ is δ-far from the maximum value $\log d$ can be bounded as follows:

$$\mathbb{P}_H\left\{S_{\boldsymbol{P}}(U\rho U^\dagger) \leq (1-\delta)\log d\right\} \leq \frac{4}{\kappa(\boldsymbol{P})} e^{-C\delta\kappa(\boldsymbol{P})^2 d\log d},$$

where $\kappa(\boldsymbol{P}) = \min_x \operatorname{tr}\{P_x\, u\} = \frac{1}{d}\min_x V_x$ and $C = \frac{1}{18\pi^3} \approx 0.0018 > 2^{-10}$.

Generic increase of OE

For an asymptotically coarse sequence $\{\boldsymbol{P}^{(d)}\}_{d\in\mathbb{N}}$ of POVM, i.e., such that $\kappa(d) = \Omega(d^{-\frac{1}{2}+\tau})$, the right-hand side of

$$\mathbb{P}_H \left\{ S_{\boldsymbol{P}}(U\rho U^\dagger) \leq (1-\delta)\log d \right\} \leq \frac{4}{\kappa(\boldsymbol{P})} e^{-C\delta\kappa(\boldsymbol{P})^2 d \log d}$$

is of order $d^{\frac{1}{2}-\tau} e^{-C\delta d^{2\tau}\log d}$, which goes to zero for any $\delta > 0$ in the limit of $d \to \infty$. Therefore, for a system of sufficiently large dimension d, it holds that

$$\mathbb{P}_H \left\{ \frac{S_{\boldsymbol{P}^{(d)}}(U\rho U^\dagger)}{\log d} > 1-\delta \right\} \approx 1 \,,$$

for any $\delta > 0$.

Generic increase of OE

Definition

Consider a sequence of systems with increasing dimension d and, in each system, a POVM $\boldsymbol{P}^{(d)} = \{P_{x_d}^{(d)}\}_{x_d}$. For each d, define $\kappa(\boldsymbol{P}^{(d)}) \equiv= \min_{x_d} \mathrm{tr}\left\{ P_{x_d}^{(d)} \ u \right\}$. We say that the sequence of POVMs $\{\boldsymbol{P}^{(d)}\}_{d\in\mathbb{N}}$ is *asymptotically coarse* whenever there exists $\tau > 0$ such that

$$\kappa(\boldsymbol{P}^{(d)}) = \Omega(d^{-\frac{1}{2}+\tau}) \ .$$

i.e., whenever $\exists M > 0$ and $\exists d_0$ such that

$$\kappa(\boldsymbol{P}^{(d)}) \geq M \cdot d^{-\frac{1}{2}+\tau} \ , \quad \forall d > d_0 \ .$$

- This means that for coarse (macro) measurements on a sufficiently large quantum system, $\mathbb{P}_H\{S_{\boldsymbol{P}}(U\rho U^\dagger) \approx \log d\} \approx 1$.
- Similar results hold in a more physically feasible pseudo-random setting (approximate 2-design).

Generic increase of OE

Definition (ε-approximate (unitary) t-design)

Fix $\varepsilon > 0$ and $t \in \mathbb{N}$. A unitary ensemble $\mathcal{E} := \{p_i, U_i\}_{i=1}^N$ is an ε-approximate (unitary) t-design in the diamond distance if

$$\left\| M_{\mathcal{E}}^{(t)} - M_H^{(t)} \right\|_\diamond \leq \frac{t!}{d^{2t}} \varepsilon \ .$$

Here, $M_{\mathcal{E}}^{(t)}(X) := \sum_{i=1}^n p_i U_i^{\otimes t} X (U_i^\dagger)^{\otimes t}$ and $M_H^{(t)}(X) := \mathbb{E}[U^{\otimes t} X (U^\dagger)^{\otimes t}]$.

Theorem (ε-approximate (unitary) t-design case)

For a unitary U sampled at random from an ε-approximate 2-design $\mathcal{E}$,

$$\mathbb{P}_{\mathcal{E}} \left\{ S_{\boldsymbol{P}}(U \rho U^\dagger) \leq (1-\delta) \log d \right\} \leq \frac{1}{\kappa(\boldsymbol{P})^3 d \log d} \frac{4(1+\varepsilon)}{\delta} \ ,$$

for any value $\delta > 0$.

Generic increase of OE

- The upper bound provided in an ε-approximate 2-design $\mathcal{E}$ is less stringent than that presented in the Haar distribution.
- This is due to the fact that the negative exponential rate in d, which was present in the Haar distribution, has been replaced by $(d \log d)^{-1}$.
- Now, in the case of ε-approximate 2-designs, we require that $\kappa(d) = \Omega(d^{-\frac{1}{3}+\tau})$ for some $\tau > 0$.
- Therefore, for any sufficiently large d, even in the case where U is sampled from an ε-approximate 2-design $\mathcal{E}$, it still holds that, for any $\delta > 0$,

$$\mathbb{P}_{\mathcal{E}} \left\{ \frac{S_{\boldsymbol{P}^{(d)}}(U\rho U^\dagger)}{\log d} > 1 - \delta \right\} \approx 1 \,.$$

Conclusion

- This research presents three methods in which observational entropy is shown to increase and reach its maximum in isolated systems undergoing a generic unitary evolution.
- Firstly, if the initial state is a macroscopic state, observational entropy will increase with probability 1.
- Secondly, if we consider unitary time evolution in Haar random for arbitrary initial states, we can show that it increases with very high probability for coarse (macroscopic) measurements on sufficiently large quantum systems.
- Finally, it was found that observational entropy increases not only in settings that are physically difficult to realise, such as Haar random, but also in more physically realisable pseudo-random settings (approximate 2-design).