Quantum Tuyman lemma in Geometric Quantization

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Outline

In this talk, I will introduce some of the joint works with Kwokwai Chan and Conan Leung on quantum Tuyman lemma in geometric quantization and its applications.

- Geometric quantization and Berezin-Toeplitz operators
- Tuyman's lemma revisited
- Kähler quantization via Kapranov's L_∞ structure
- Quantum Tuyman's lemma and applications

Mathematical thoery of quantization

The phase space of a classical mechanical system is described geometrically as a symplectic manifold. The mathematical formulation of quantum mechanics is in terms of operators on Hilbert spaces satisfying the Dirac axioms. There are two schemes for quantization on symplectic manifolds: one is deformation quantization which focuses on the algebra of operators, and the other is geometric quantization which focuses on the Hilbert space of the quantum system.

The construction of geometric quantization (Hilbert spaces) depends on a choice of polarization. There are two types of polarizations: real and complex polarizations. We will focus on complex polarizations, which is equivalent to a complex structure on the phase space making it a Kähler manifold.

Geometric quantization and Toeplitz operators

On a Kähler manifold X, the Hilbert space (of level k) of its geometric quantization is defined as

$$\mathcal{H}_k := H^0_{\bar\partial}(X, L^{\otimes k}).$$

Here L denotes the prequantum line bundle on X.

On these Hilbert spaces $\mathcal{H}_k$, we can define Toeplitz operators associated to a smooth function $f \in C^\infty(X)$ as $T_{f,k} = \Pi_k \circ m_f$. Here m_f denotes the multiplication by a smooth function $f \in C^\infty(X)$, and Π_k denotes the orthogonal projection from smooth sections to holomorphic sections.

Toeplitz operators and deformation quantization

For two functions f and g, the compositions $T_{f,k} \circ T_{g,k}$ has the following asymptotic property as $k \to \infty$:

$$T_{f,k} \circ T_{g,k} \sim T_{fg,k} + \sum_{i \geq 1} \left(\frac{1}{k}\right)^i \cdot T_{C_i(f,g),k}.$$

This gives rise to a *deformation quantization* of smooth functions on X by turning $1/k$ into the formal variable $\hbar$.

Deformation quantization

Deformation quantization is an associative but non-commutative deformation of the algebra of smooth functions on symplectic manifolds.

Definition

Let (M, ω) be a symplectic manifold, then a deformation quantization of M is an associative product $*$ on $C^\infty(M)[[\hbar]]$ such that

$$f * g = f \cdot g + \sum_{i \geq 1} \hbar^i C_i(f, g),$$

where C_i's are bi-differential operators, with $C_1(f, g) - C_1(g, f) = \{f, g\}$.

Tuyman's lemma revisited

Recall that the Tuyman's lemma is explicitly:

Theorem (Tuyman)

For every smooth function $f \in C^\infty(X)$*, and any two holomorphic sections* $s_1, s_2 \in H^0_{\bar\partial}(X, L^{\otimes k})$*, there is the following equality*

$$\langle \nabla_{X_f} s_1 + \frac{1}{k}\Delta f \cdot s_1, s_2\rangle = 0.$$

Here X_f *denotes the vector field associated to* f*, and* Δ *is the Laplacian on* X*.*

Since the holomorphic section s_2 is arbitrary, this implies that $\left(\nabla_{X_f} + \frac{1}{k}\Delta f\right) s$ lies in the orthognoal complement of the Hilbert space $H^0(X, L^{\otimes k}) \subset C^\infty(X, L^{\otimes k})$.

Tuyman's lemma revisited

We can interprete Tuyman's lemma as follows: for every smooth function f, we obtain an associated differential operator

$$\nabla_{X_f} + \frac{1}{k}\Delta f : H^0(X, L^{\otimes k}) \to C^\infty(X, L^{\otimes k}),$$

whose images actually lie the in the complement of the subspace of holomorphic sections.

Question

Are there other differential operators associated to smooth functions, such that they share the same property as in Tuyman's lemma?

Quantization of the L_∞ structure

In 1996, Kapranov discovered an L_∞ structure on Kähler manifolds in his study of Rozansky-Witten theory. The main ingredient of this strucutre is a cochain complex whose cohomology is isomorphic to holomorphic functions, which gives an equivalent description of the complex structure (i.e., complex polarization) on X.

Theorem (Chan-Leung-L)

There exists a Fedosov connection D on X which is a quantum extension of Kapranov's L_∞ structure on Kähler manifolds.

According to the first results, this Fedosov connection induces a BV quantization of the Kähler manifold X. This quantization encodes not only the symplectic geometry (i.e., phase space) but also the complex geometry.

Fedosov connection on Kähler manifolds

We briefly describe the above Fedosov connection, i.e., quantum master equation. The idea behind Fedosov's original approach to deformation quantization is as follows: on the Weyl bundle $\mathcal{W}_{X,\mathbb{C}} := \widehat{\mathrm{Sym}}\, TX^*_{\mathbb{C}}$ over the phase space X, there exists a fiberwise star product $\star$ which describes the local picture of quantization on $\mathbb{C}^n$, and a (flat) Fedosov connection gives the gluing data for these local quantizations which is of the following form:

$$\nabla + \frac{1}{\hbar}[\gamma, -]_\star.$$

Here $[-,-]_\star$ denotes the bracket associated to the Wick product. The flat sections under this connections are isomorphic to the quantum observables $C^\infty(X)[[\hbar]]$.

Bargmann-Fock bundle

We follow the same line of thought and give a construction of Bargmann-Fock bundle on Kähler manifolds, starting with the fiberwise Bargmann-Fock action of the Weyl bundle $\mathcal{W}_{X,\mathbb{C}}$ on $\mathcal{W}_X = \widehat{\mathrm{Sym}}\, TX^*$, which we denote by $\circledast$.

By twisting $\mathcal{W}_X$ with the tensor power of the prequantum line bundle $L^{1/\hbar}$, we obtain the Bargmann-Fock bundle $\mathcal{F}_X$ as a module sheaf over the Weyl bundle with respect to the fiberwise star product. There exists a Fedosov connection (i.e., quantum master equation) on the Fock bundle of the following form:

$$D = \nabla + \frac{1}{\hbar}\gamma \circledast -.$$

Bargmann-Fock bundle

Theorem (Chan-Leung-L)

- *The Fedosov connections on the Weyl bundle and Fock bundle are compatible in the sense that* $D(\alpha \circledast s) = D(\alpha) \circledast s + \alpha \circledast D(s)$.
- *Taking the evaluation* $\hbar = 1/k$, *the flat sections of the Fock bundle is isomorphic to* $H^0_{\bar{\partial}}(X, L^{\otimes k})$.

This result implies that we obtain a quantization scheme which involves both quantum observables and geometric quantization (Hilbert spaces) as modules, using the language of formal geometry and homological algebra.

A weight on the Weyl bundle

We will focus on the Berezin-Toeplitz deformation quantization, and will denote the corresponding Fedosov connection by D_{BT} (here the subscript BT is short for Berezin-Toeplitz), which induces a cochain complex

$$\left(\mathcal{A}_X^\bullet \otimes \mathcal{W}_{X,\mathbb{C}}, D_{BT}\right).$$

We will call this the *Fedosov complex*.

Definition

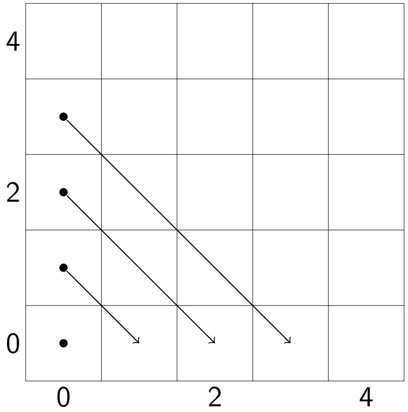
We define a weight on the formal Weyl bundle $\mathcal{W}_{X,\mathbb{C}}[[\hbar]]$ by assigning weights on generators:

$$|\hbar| = 1, \quad |\bar{y}^j| = 1, \quad |y^i| = 0.$$

The fiberwise Wick product $\star$ on $\mathcal{W}_{X,\mathbb{C}}$ is compatible with this weight.

A weight on the Weyl bundle

We have the following picture illustrating the weights: here the horizontal direction denotes the power of $\hbar$, and the vertical direction denotes the polynomial degree in the anti-holomorphic Weyl bundle $\overline{\mathcal{W}}_X$.



Differential operators associated to functions

One of the main result in the theory of Fedosov quantization is that given any smooth function $f \in C^\infty(X)$, there is a uniquely determined flat section O_f of the Weyl bundle such that its symbol $\sigma(O_f) = f$.

This is also generalized to the Bargman-Fock module case: there is the following one-to-one correspondence via the symbol map:

$$\Gamma^{\mathrm{flat}}(X, \mathcal{F}_k) \cong H^0(X, L^{\otimes k}).$$

More explicitly, given any holomorphic section $s \in H^0(X, L^{\otimes k})$, there is an associated flat section O_s.

Differential operators associated to functions

Using the weight we defined on the Weyl bundle $\mathcal{W}_{X,\mathbb{C}}$, we can decompose a flat section O_f as

$$O_f = (O_f)_0 + (O_f)_1 + \cdots .$$

We are now ready to define a series of operators via the Fedosov quantization method as follows: for each $m \geq 0$, we let

$$\gamma_{f,m}(s) := \sigma\left((O_f) \circledast_k O_s\right).$$

The output is in general only a smooth section of $L^{\otimes k}$. Since at each $x_0 \in X$, the flat sections O_f and O_s only depends on the infinite jet (Taylor expansion) of f and s respectively, we obtain a differential operator $\gamma_{f,m} : H^0(X, L^{\otimes k})$.

Differential operators associated to functions

Here we give an example. A simple computation shows that for weight $m = 1$ case, there is

Lemma

The differential operator $\gamma_{f,1}$ is exactly the one in Tuyman's lemma:

$$\gamma_{f,1} = \nabla_{X_f} + \frac{1}{k}\Delta f.$$

Quantum Tuyman lemma

Thus the classical Tuyman lemma can written as the following orthogonality relation:

$$\langle \gamma_{f,1}(s_1), s_2 \rangle = 0,$$

for any holomorphic sections $s_1, s_2 \in H^0(X, L^{\otimes k})$. It is then not surprising that for all weights $m \geq 1$, there is a similar *orthogonality relation*. This is what we call the *Quantum Tuyman lemma*.

Theorem (Chan-Leung-L)

For every weight $m \geq 1$, we have the following vanishing for any holomorphic sections $s_1, s_2 \in H^0(X, L^{\otimes k})$:

$$\langle \gamma_{f,m}(s_1), s_2 \rangle = 0.$$

Equivalently, $\gamma_{f,m}(s_1)$ always lives in the orthogonal complement of $H^0(X, L^{\otimes k})$.

Quantum Tuyman lemma

I will briefly describe the proof of this theorem. Recall that the differential operators $\gamma_{f,m}(s)$ is defined using the flat sections associated to both $f \in C^\infty(X)$ and $s \in H^0(X, L^{\otimes k})$:

$$\sigma((O_f)_m \circledast O_s).$$

Using the fact that both O_f and O_s are flat under the Fedosov connections, we can show that the following top degree differential form on X

$$\langle \gamma_{f,m}(s_1), s_2 \rangle \cdot \omega^n$$

is actually exact. Thus the Stoke's formula implies the desired vanishing.

Application: Quantum symmetry

Infinitesimally, a quantum symmetry is a vector field V on X which preserves both the symplectic and complex structure of X. This type of vector field can be lifted to an action on smooth sections of (tensor power of) the pre-quantum line bundle $L^{\otimes k}$, which preserves the sub-space consisting of holomorphic sections. This action is given by the following explicit formula:

$$\mathcal{L}_V(s) = \nabla_V + \frac{1}{k}\mu(V).$$

Here $f = \mu(V)$ denotes the moment map associated to the vector field V. By using the original Tuyman's lemma, we can show that the infinitesimal action $\mathcal{L}_V$ can be identified with a Berezin-Toeplitz operator.

Quantum symmetry

More precisely, Tuyman's lemma applied to the moment map $f = \mu(V)$ implies that

$$\Pi_k \circ (\nabla_{X_f} + \frac{1}{k}\Delta f)(s) = \Pi_k \circ (\nabla_V + \frac{1}{k}\Delta f)(s) = 0.$$

Here Π_k denotes the orthogonal projection to the subspace of holomorphic sections. And there is

$$\begin{aligned}\mathcal{L}_V(s) =& \Pi_k \circ \mathcal{L}_V(s) = \Pi_k \circ (\nabla_V + \frac{1}{k}\mu(V))(s)\\ =& \Pi_k \circ (-\frac{1}{k}\Delta f + \frac{1}{k}f)(s).\end{aligned}$$

The right hand side of the last equality is exactly a Berezin-Toeplitz operator.

Quantizable functions and differential operators

As another application of the Quantum Tuyman Lemma, we will give an answer to the following question:

Question

When is the Berezin-Toeplitz operator $T_{f,k}$ associated to a smooth function f a holomorphic differential operator?

We have the following theorem:

Theorem (Chan-Leung-L)

For a smooth function $f \in C^\infty(X)$, its associated Berezin-Toeplitz operator $T_{f,k}$ is a holomorphic differential operator if and only if f is a quantizable function.

Quantizable functions and differential operators

Here I will briefly describe the proof of this theorem. Given a quantizable function f, the associated flat section O_f are denoted by the stars in the following picture.

2	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	$*$	$*$	$*$	$\cdots$
0	f	0	0	$\cdots$
	0		2	

Quantizable functions and differential operators

On one hand, the Berezin-Toeplitz operator $T_{f,k}$ is defined using the terms in the first row, i.e., the function f itself. On the other hand, the corresponding holomorphic differential operator is defined using the Bargmann-Fock action the Fedosov construction, in which all the terms in O_f is needed.

However, by the Quantum Tuyman Lemma, all the terms in O_f which is NOT in the first row will only contribute terms which is orthogonal to the subspace $\mathcal{H}_k \subset C^\infty(X, L^{\otimes k})$. And the theorem follows from the following equality:

$$f \cdot s + \text{orthogonal complement terms} = \sigma\left(O_f \circledast_k O_s\right) \in H^0(X, L^{\otimes k}),$$

where the right hand side is exactly $T_{f,k}(s)$.

Further questions

1. Now that differential operators can be identified with quantizable functions, are there symplectic reductions of G-invariant operators on $M//G$? What happens to their correlation functions?
2. Are there higher dimensional analogues of this picture? For instance, in two dimensional field theories, what are the "quantizable function" and quantum symmetries in chiral differential operators?

Thank You!