Time-Efficient Quantum Entropy Estimator via Samplizer

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Entropy

Probability distribution p		Quantum state ρ
$H(p) = -\sum_j p_j \ln(p_j)$ Shannon entropy		$S(\rho) = -\mathrm{tr}(\rho \ln(\rho))$ Von Neumann entropy
	$\xrightarrow{\text{generalize}}$	
$H_\alpha(p) = \frac{1}{1-\alpha} \ln(\sum_j p_j)$ Rényi entropy		$S_\alpha(\rho) = \frac{1}{1-\alpha} \ln(\mathrm{tr}(\rho^\alpha))$ Quantum Rényi entropy

Classical Entropy Estimators

- Shannon entropy estimator.
  [Valiant and Valiant, STOC 2011, NIPS 2013, *J. ACM* 2017]
  [Jiao, Venkat, Han, and Weissman, *IEEE Trans. Inf. Theory* 2015, 2017]
  [Wu and Yang, *IEEE Trans. Inf. Theory* 2016]
  - Sample complexity: $\Theta(N/\log(N))$ for N-dimensional distributions.
- α-Rényi entropy estimator.
  [Acharya, Orlitsky, Suresh, and Tyagi, *IEEE Trans. Inf. Theory* 2017]
  - Sample complexity: $\widetilde{O}(N^{1/\alpha})$ for $0 < \alpha < 1$.
  - Sample complexity: $\widetilde{O}(N)$ for $\alpha > 1$.

ALL these estimators have time complexity *linear* in the sample complexity.

Quantum Entropy Estimators

- Sample Access:
  Given access to identical copies of the quantum state.
- Query Access:
  Given access to the state-preparation circuit of the quantum state.

Quantum Entropy Estimators I: Sample Access

- Von Neumann entropy estimator.
  [Acharya, Issa, Shende, and Wagner, ISIT 2019, *IEEE J. Sel. Areas Inf. Theory* 2020]
  [Bavarian, Mehraban, and Wright, 2016]
  - Sample complexity: $O(N^2)$, Time complexity: $\widetilde{O}(N^6)$.
- Quantum α-Rényi entropy estimator.
  [Acharya, Issa, Shende, and Wagner, ISIT 2019, *IEEE J. Sel. Areas Inf. Theory* 2020]
  - Sample complexity: $O(N^{2/\alpha})$, Time complexity: $\widetilde{O}(N^{6/\alpha})$ for $0 < \alpha < 1$.
  - Sample complexity: $O(N^2)$, Time complexity: $\widetilde{O}(N^6)$ for $\alpha > 1$.

Quantum Entropy Estimators II: Query Access

- Von Neumann entropy estimator.
  [Gilyén and Li, ITCS 2020]
  [Gur, Hsieh and Subramanian, QIP 2022]
  [Wang, Guan, Liu, Zhang, and Ying, *IEEE Trans. Inf. Theory* 2024]
  - Query complexity: $\widetilde{O}(N)$.
- Quantum α-Rényi entropy estimator.
  [Subramanian and Hsieh, *Phys. Rev. A* 2021]
  [Wang, Guan, Liu, Zhang, and Ying, *IEEE Trans. Inf. Theory* 2024]
  [Wang, Zhang, and Li, *IEEE Trans. Inf. Theory* 2024]
  - Query complexity: $\widetilde{O}(N^{\frac{1}{2\alpha}+\frac{1}{2}})$ for $0 < \alpha < 1$.
  - Query complexity: $\widetilde{O}(N)$ for $\alpha > 1$.

Motivation

There is a cubic gap between the sample complexity and the time complexity of the known quantum entropy estimators with sample access.

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Motivation

There is a cubic gap between the sample complexity and the time complexity of the known quantum entropy estimators with sample access.

Question:

Can we estimate quantum entropy

with time complexity linear in the sample complexity?

Main Results

We propose new quantum entropy estimators based on an algorithmic tool *samplizer*.

Table: Sample and time complexities for entropy estimation of quantum states.

	Reference	$0 < \alpha < 1$	$\alpha = 1$ (von Neumann)	$\alpha > 1$
Upper Bounds	[AISW20]	$O(N^{2/\alpha})$ samples $\widetilde{O}(N^{6/\alpha})$ time	$O(N^2)$ samples $\widetilde{O}(N^6)$ time	$O(N^2)$ samples $\widetilde{O}(N^6)$ time
	This work	$\widetilde{O}(N^{4/\alpha-2})$ samples $\widetilde{O}(N^{4/\alpha-2})$ time	$\widetilde{O}(N^2)$ samples $\widetilde{O}(N^2)$ time	$\widetilde{O}(N^{4-2/\alpha})$ samples $\widetilde{O}(N^{4-2/\alpha})$ time
Lower Bounds	[AISW20]	$\Omega(N^{1+1/\alpha})$ (EYD)	$\Omega(N^2)$ (EYD)	$\Omega(N^2)$ (EYD)
	This work	$\Omega(N + N^{1/\alpha-1})$	$\Omega(N)$	$\Omega(N)$

The lower bounds tagged EYD only hold for those algorithms based on Empirical Young Diagram algorithms.

IDEA in One Sentence

Design quantum sample-access algorithms
by imitating quantum query-access algorithms.

Intuition: Quantum query-access algorithms usually have time complexity linear (up to polylogarithmic factors) in the query complexity.

A Simple Quantum Query Algorithm

Consider a quantum query algorithm for estimating von Neumann entropy based on quantum singular value transformation (QSVT) [Gilyén, Su, Low, and Wiebe, STOC 2019].

The key step:

$$U = \begin{bmatrix} \rho & * \\ * & * \end{bmatrix} \xrightarrow[-\ln(\cdot)]{\text{QSVT}} \begin{bmatrix} -\ln(\rho) & * \\ * & * \end{bmatrix} =: U'$$

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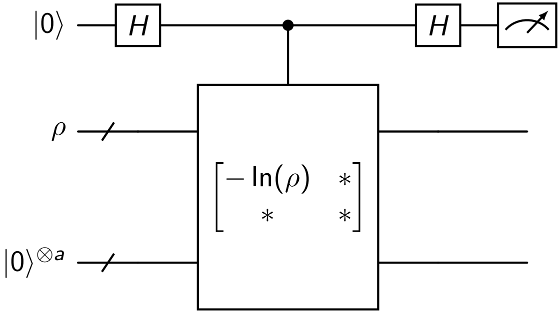
Then, by applying the Hadamard test [Aharonov, Jones, and Landau, STOC 2006, *Algorithmica* 2009] on U' and ρ, we can estimate $\mathrm{tr}(-\rho\ln(\rho))$.

A Simple Quantum Query Algorithm, Visualized



Samplizer

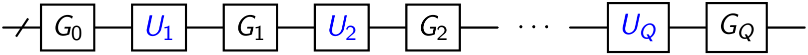
We propose a useful tool — **samplizer** that can "samplize" a quantum query algorithm to a quantum sample algorithm with similar behavior.

Quantum Query Algorithms

A quantum query algorithm with access to quantum oracle U is described by a quantum circuit family $C = \{C[U]\}$ with

$$C[U] = G_Q U_Q \dots G_2 U_2 G_1 U_1 G_0,$$

where U_i is either (controlled-)U or (controlled-)$U^\dagger$ and G_i consists of one- and two-qubit quantum gates.



Here, Q is the query complexity of quantum query algorithm C.

Quantum Query Algorithms

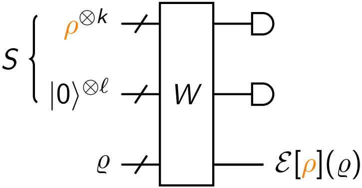
Examples:

- Quantum search
  [Grover, STOC 1996]
- Element distinctness
  [Amb07, FOCS 2004, *SIAM J. Comput.* 2007]
- Solving systems of linear equations
  [Harrow, Hassidim, and Lloyd, *Phys. Rev. Lett.* 2009]
- Hamiltonian simulation
  [Low and Chuang, *Quantum* 2019]
- ...

Quantum Sample Algorithms

A quantum sample algorithm with access to independent samples of a quantum state ρ is described by a quantum channel family $\mathcal{E} = \{\mathcal{E}[\rho]\}$ implemented by a quantum circuit W such that

$$\mathcal{E}[\rho](\varrho) = \mathrm{tr}_S\left(W \left(\underbrace{\rho^{\otimes k} \otimes |0\rangle\langle 0|^{\otimes \ell}}_{S} \otimes \varrho \right) W^\dagger \right).$$



Here, k is the sample complexity of quantum sample algorithm $\mathcal{E}$.

Quantum Sample Algorithms

Examples:

- Quantum state tomography
  [Haah, Harrow, Ji, Wu, and Yu, STOC 2016, *IEEE Trans. Inf. Theory* 2017]
  [O'Donnell and Wright, STOC 2016]
- Quantum state certification
  [Bădescu, O'Donnell and Wright, STOC 2019]
- Shadow tomography of quantum states
  [Aaronson, STOC 2018, *SIAM J. Comput.* 2020]
  [Huang, Kueng, and Preskill, *Nat. Phys.* 2020]

Samplizer

Definition 1 (Samplizer)

A samplizer $\mathsf{Samplize}_*\langle * \rangle$ is a converter from a quantum query algorithm to a quantum sample algorithm with the following property:
For $\delta > 0$, quantum query algorithm $C = \{C[U]\}$, and quantum state ρ, there exists a unitary operator U_ρ that is a block-encoding of ρ such that

$$\left\| \mathsf{Samplize}_\delta\langle C \rangle[\rho] - C[U_\rho] \right\|_\diamond \leq \delta,$$

where $\|\cdot\|_\diamond$ denotes the diamond norm between quantum channels.

Here, U is a block-encoding of A if the matrix A is in the upper left corner in the matrix representation of U (in the computational basis), i.e.,

$$U = \begin{bmatrix} A & * \\ * & * \end{bmatrix}.$$

Samplizer

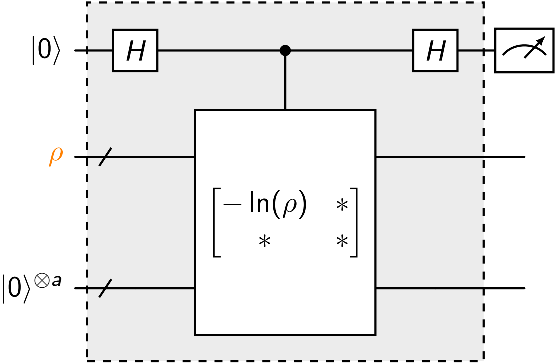
Theorem 2 (Optimal samplizer)

There is an optimal samplizer $\mathsf{Samplize}_*\langle * \rangle$ *such that for any* $\delta > 0$ *and quantum query algorithm* C *with query complexity* Q*, the quantum sample algorithm* $\mathsf{Samplize}_\delta\langle C \rangle$ *has sample complexity* $\widetilde{\Theta}(Q^2/\delta)$*.*

Quantum query algorithm		Quantum sample algorithm
C		$\mathsf{Samplize}_\delta\langle C \rangle$
Quantum query complexity	$\xrightarrow{\text{Samplizer}}$	Quantum sample complexity
Q		$\widetilde{\Theta}(Q^2/\delta)$

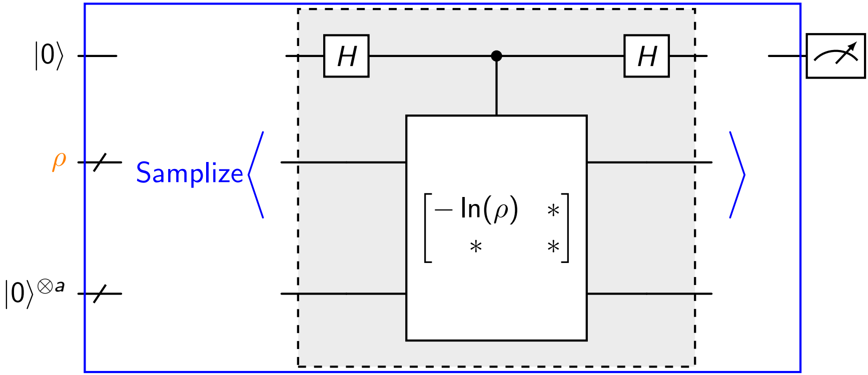
Time-Efficient Von Neumann Entropy Estimator

Recall the simple quantum query algorithm:



Time-Efficient Von Neumann Entropy Estimator

The main part of the quantum circuit can be *samplized*:



Time-Efficient Von Neumann Entropy Estimator

Algorithm 1 `estimate_von_Neumann_main`(ε, δ) — *quantum sample algorithm*

Resources: Access to independent samples of N-dimensional quantum state ρ of rank r.
Input: $\varepsilon \in (0,1)$ and $\delta \in (0,1)$.
Output: $\widetilde{S}$ such that $|\widetilde{S} - S(\rho)| \le \varepsilon$ with probability $\ge 1-\delta$.

function `von_Neumann_subroutine`$(\delta_p, \varepsilon_p, \delta_Q)$ — *quantum query algorithm*
  Resources: Unitary oracle U_A that is a block-encoding of A.
  Let $p(x)$ be a polynomial of degree $d_p = O\left(\frac{1}{\delta_p}\log\left(\frac{1}{\varepsilon_p}\right)\right)$ such that $|p(x)| \le \frac{1}{2}$ for $x \in [-1,1]$ and $\left|p(x) - \frac{\ln(1/x)}{4\ln(2/\delta_p)}\right| \le \varepsilon_p$ for $x \in [\delta_p, 1]$.
  Construct unitary operator $U_{p(A)}$ that is a $(1, a, \delta_Q)$-block-encoding of $p(A)$, using $O(d_p)$ queries to U_A.
  return $U_{p(A)}$. $\leftarrow$ a unitary block-encoding of $p(A) \propto\!\!\!\!\sim -\ln(A)$ using $\widetilde{O}(1/\delta_p)$ queries to U_A.
end function
$\delta_p \leftarrow \frac{\varepsilon}{128r\ln(32r/\varepsilon)}$, $\varepsilon_p \leftarrow \frac{\varepsilon}{32\ln(2/\delta_p)}$, $\delta_Q \leftarrow \frac{\varepsilon}{32r\ln(2/\delta_p)}$, $\delta_a \leftarrow \frac{\varepsilon}{64\ln(2/\delta_p)}$, $\varepsilon_H \leftarrow \delta_a$, $k \leftarrow \left\lceil \frac{1}{2\varepsilon_H^2}\ln\left(\frac{2}{\delta}\right)\right\rceil$.
for $i = 1 \ldots k$ **do**
  Perform the Hadamard test on $\mathsf{Samplize}_{\delta_a}\langle$`von_Neumann_subroutine`$(\delta_p, \varepsilon_p, \delta_Q)\rangle[\rho]$ (δ_a-close to a unitary block-encoding of $-\ln(\rho)$ using $\widetilde{O}(1/\delta_p^2\delta_a)$ samples of ρ) and ρ. Let $X_i \in \{0,1\}$ be the outcome.
end for
$\widetilde{S} \leftarrow 4(2\sum_{i\in[k]} X_i/k - 1)\ln(2/\delta_p) - \ln(2) \approx -\mathrm{tr}(\rho\ln(\rho)) = S(\rho)$.
return $\widetilde{S}$.

Time-Efficient Quantum Entropy Estimators

Sample (and time) complexity of Algorithm 1:

$$k \cdot \widetilde{O}(1/\delta_p^2 \delta_a) = \widetilde{O}(N^2/\varepsilon^5).$$

Theorem 3

Algorithm 1 estimates the von Neumann entropy to precision ε with sample and time complexity $\widetilde{O}(N^2/\varepsilon^5)$.

Moreover, inspired by the quantum query algorithm for estimating the α-Rényi entropy of probability distributions in [Li and Wu, *IEEE Trans. Inf. Theory* 2019], we further have:

Theorem 4

We can estimate the quantum α-Rényi entropy of quantum states to precision ε with sample and time complexity $\widetilde{O}(N^{4/\alpha-2}/\varepsilon^{1+4/\alpha})$ for $0 < \alpha < 1$ and $\widetilde{O}(N^{4-2/\alpha}/\varepsilon^{3+2/\alpha})$ for $\alpha > 1$.

Implementation of Samplizer

Step 1. Density matrix exponentiation.
[Lloyd, Mohseni, and Rebentrost, *Nat. Phys.* 2014]
[Kimmel, Lin, Low, Ozols, and Yoder, *npj Quantum Inf.* 2017]

Theorem 5 (Density matrix exponentiation)

We can implement a quantum channel $\mathcal{E}$ such that

$$\left\|\mathcal{E} - e^{-i\rho t}\right\|_\diamond \leq \delta,$$

using $O(t^2/\delta)$ samples of ρ.

Implementation of Samplizer

Step 2. Quantum singular value transformation.
[Gilyén, Su, Low, and Wiebe, STOC 2019]
By implementing the logarithm of unitaries, we can transform

$$e^{-i\rho} \to \begin{bmatrix} \rho & * \\ * & * \end{bmatrix}$$

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Theorem 6 ([Gilyén and Poremba, TQC 2022])

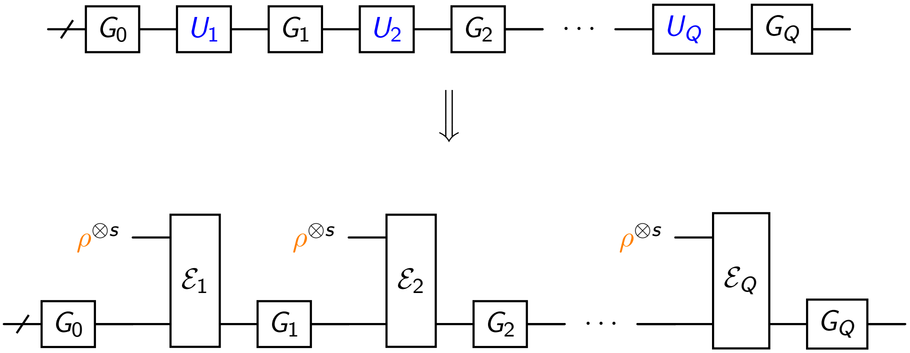
We can implement a quantum channel $\mathcal{E}$ such that there is a unitary operator U that is a block-encoding of ρ and

$$\|\mathcal{E} - \mathcal{U}\|_\diamond \le \delta,$$

using $\widetilde{O}(1/\delta)$ samples of ρ.

Implementation of Samplizer

Step 3. By generalizing the quantum sample-to-query lifting theorem [Wang and Zhang, 2023], we have the following construction.



$$s = \widetilde{O}(Q/\delta)$$

Optimality of Samplizer

We can also show that the implementation of the samplizer is optimal only up to polylogarithmic factors, by combining two prior results:

- Sample lower bound for sample-based Hamiltonian simulation. [Kimmel, Lin, Low, Ozols, and Yoder, *npj Quantum Inf.* 2017]
- Optimal quantum query algorithm for Hamiltonian simulation. [Gilyén, Su, Low, and Wiebe, STOC 2019]

Theorem 7

Any samplizer requires sample complexity $\Omega(Q^2/\delta)$.

Open Problems

- The sample complexity of the samplizer is shown to be $\widetilde{\Theta}(Q^2/\delta)$. Can we make it tighter, improving the logarithmic factors?

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- All of the existing quantum estimators for von Neumann entropy have sample complexity $\widetilde{O}(N^2)$, while the best known sample lower bound is only $\Omega(N)$ as suggested by [O'Donnell and Wright, STOC 2015, *Comm. Math. Phys.* 2021].
  Can we show a matching lower bound $\Omega(N^2)$?

Open Problems

- The sample complexity of the samplizer is shown to be $\widetilde{\Theta}(Q^2/\delta)$. Can we make it tighter, improving the logarithmic factors?
- All of the existing quantum estimators for von Neumann entropy have sample complexity $\widetilde{O}(N^2)$, while the best known sample lower bound is only $\Omega(N)$ as suggested by [O'Donnell and Wright, STOC 2015, *Comm. Math. Phys.* 2021].
  Can we show a matching lower bound $\Omega(N^2)$?
- Can we obtain new quantum sample algorithms for other problems of interest through the samplizer?

Thank You!