2-Morita Equivalent Condensable Algebras and Domain walls in 2d Topological Orders

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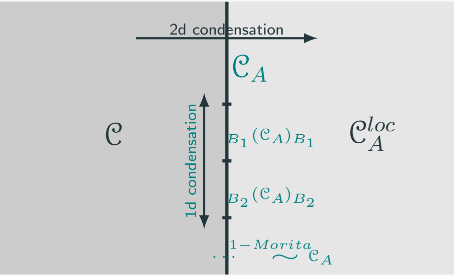
Based on [arXiv: 2403.19779], a joint work with Holiverse Yang

Primer [Kitaev06, Bais-Slingerland-Haaker09, Davydov10, Kitaev-Kong12, Davydov-Müger-Nikshych-Ostrik13, Kong14]

- A 2d anomaly-free topological order can be described by a unitary modular tensor category (MTC) $\mathcal{C}$ (with a central charge c).
- A 2d condensable (indecomposable commutative separable) algebra $A \in \mathcal{C}$ can be condensed to create new topological order, described by the category $\mathcal{C}_A^{loc}$.
- Anyons that are confined from going to the condensed phase form a 1d domain wall described by a fusion category $\mathcal{C}_A$.
- A 1d condensable algebra $B \in \mathcal{C}_A$ can be condensed to create another domain wall ${}_B(\mathcal{C}_A)_B$.

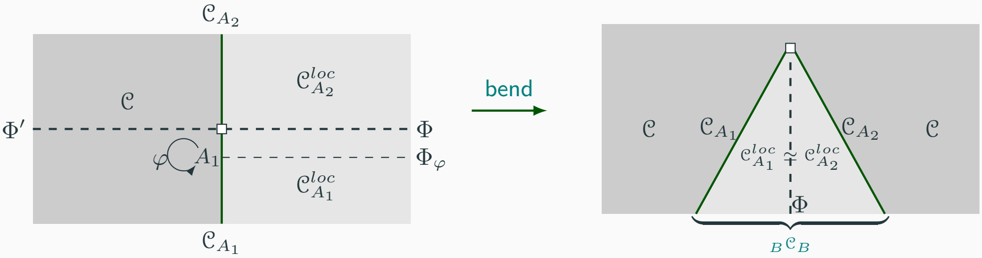


$$\mathcal{C}_{A_1}^{loc} \simeq \mathcal{C}_{A_2}^{loc}$$

$$\Updownarrow$$

$$A_1 \overset{2-Morita}{\sim} A_2$$

Motivation: classify 2-Morita equivalent A_i



Invertible domain walls can appear in a 2d topological order, which are classified by the group $\mathcal{A}\mathrm{ut}_{\otimes}^{\mathrm{br}}(\mathcal{C})$ of braided auto-equivalence. In the condensed phase we distinguish two kinds of invertible walls:

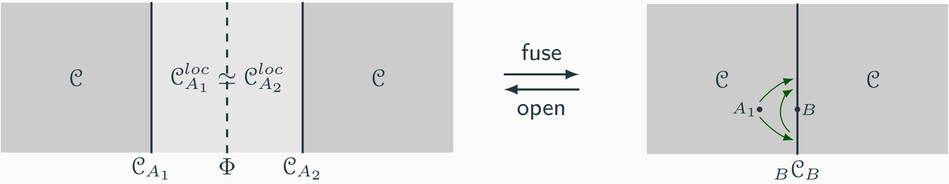
- Φ is induced from the auto-equivalences Φ' in the original phase;
- Φ_φ is induced from the algebra automorphisms of 2d condensable algebra A_1.

We denote the first situation by $A_1 \overset{2-Morita}{\underset{\phi}{\sim}} A_2$ in which $\phi : \mathcal{C}_{A_1}^{loc} \xrightarrow{\sim} \mathcal{C}_{A_2}^{loc}$ serves as an interchange of anyons between $\mathcal{C}_{A_1}^{loc}$ and $\mathcal{C}_{A_2}^{loc}$. All gapped domain walls (1 codimensional defects) within a 2d topological order $\mathcal{C}$ can be classified by (A_1, A_2, ϕ) in the sense that $\mathcal{C}_{A_1} \boxtimes_{\mathcal{C}_{A_1}^{loc}} \Phi \boxtimes_{\mathcal{C}_{A_2}^{loc}} \mathcal{C}_{A_2} \simeq {}_B\mathcal{C}_B$.

$$\begin{array}{c} B \\ \Updownarrow \\ Z_l(B) \simeq A_1 \overset{2-Morita}{\underset{\phi}{\frown}} A_2 \simeq Z_r(B) \end{array}$$

Theorem (Based on [Fröhlich-Fuchs-Runkel-Schweigert 06])

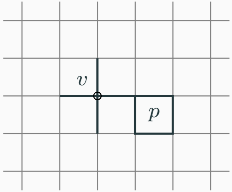
The triples (A_1, A_2, ϕ) in $\mathcal{C}$ are classified by (1-Morita classes of) indecomposable separable algebras B_i in $\mathcal{C}$, in which $A_1 \cong Z_l(B_i)$ and $A_2 \cong Z_r(B_i)$ represent the left and right centers of B_i.



Theorem (Based on [Davydov-Nikshych-Ostrik 12])

We have ${}_B\mathcal{C}_B \simeq \mathcal{C}_{Z_l(B)} \boxtimes_{\mathcal{C}^{loc}_{Z_l(B)}} \Phi \boxtimes_{\mathcal{C}^{loc}_{Z_r(B)}} \mathcal{C}_{Z_r(B)}$ where Φ is given by an equivalence of MTCs $\mathcal{C}^{loc}_{Z_l(B)} \xrightarrow{\sim} \mathcal{C}^{loc}_{Z_r(B)}$.

Example: 2d Toric code model $\mathcal{TC} := \mathfrak{Z}(\mathrm{Vec}_{\mathbb{Z}_2})$ $\mathsf{H} := \sum_v (1 - \mathsf{A}_v) + \sum_p (1 - \mathsf{B}_p)$



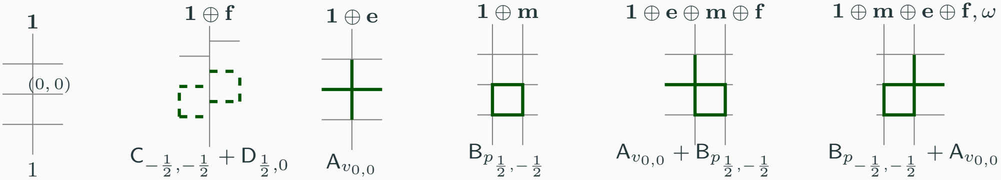
Eg, we can add a local trap $\mathsf{A}_{v_{0,0}} + \mathsf{B}_{p_{\frac{1}{2},-\frac{1}{2}}}$ to the original Hamiltonian:

$$\mathsf{H} + \mathsf{A}_{v_{0,0}} + \mathsf{B}_{p_{\frac{1}{2},-\frac{1}{2}}} = \sum_{v \neq v_{0,0}} (1 - \mathsf{A}_v) + \sum_{p \neq p_{\frac{1}{2},-\frac{1}{2}}} (1 - \mathsf{B}_p) + 2$$

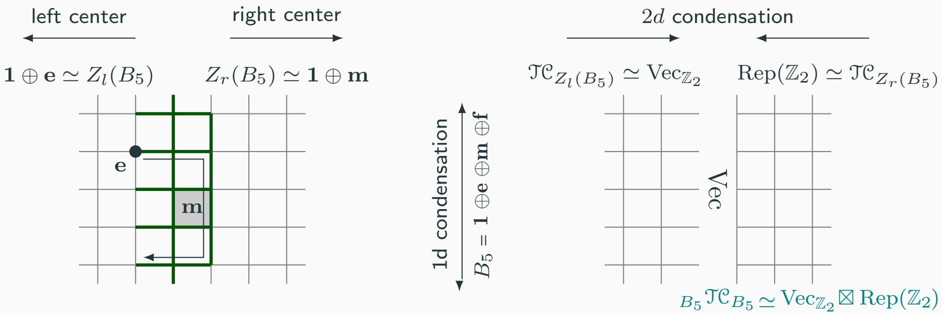
The new ground state subspace of is 4-fold degenerate, which can be distinguished by the eigenvalues of $\mathsf{A}_{v_{0,0}} = \pm 1$ and $\mathsf{B}_{p_{\frac{1}{2},-\frac{1}{2}}} = \pm 1$

The topological excitation generated by the local trap is $\mathbf{1} \oplus \mathbf{e} \oplus \mathbf{m} \oplus \mathbf{f}$.

Lattice realizations of 1d condensable algebra B_i locally.



Condensing B_i in $\mathcal{TC}$ is equivalent to removing these thick edges for all k along the neighborhood of column 0.



$$\mathsf{H}_{wall}=\mathsf{H}+\sum_k \mathsf{A}_{v_{0,k}}+\sum_k \mathsf{B}_{p_{\frac{1}{2},k-\frac{1}{2}}}=\sum_{v\neq v_{0,k}}(1-\mathsf{A}_v)+\sum_{p\neq p_{\frac{1}{2},k-\frac{1}{2}}}(1-\mathsf{B}_p)+2N$$

$B_5=\mathbf{1}\oplus\mathbf{e}\oplus\mathbf{m}\oplus\mathbf{f}$ located on the domain wall can not expand itself freely into the 2d bulk. However, its sub-algebra $\mathbf{1}\oplus\mathbf{e}/\mathbf{1}\oplus\mathbf{m}$ can be expanded to the left/right bulk. The half braiding $\beta_{\mathbf{m},\mathbf{e}}=\mathrm{Id}$ happening in the left side of the wall is trivial, however, $\mathbf{1}\oplus\mathbf{e}$ is blocked from going to the right bulk due to the non-trivial braiding $\beta_{\mathbf{e},\mathbf{m}}=-\mathrm{Id}$ in $\mathcal{TC}$. Similarly, $\mathbf{1}\oplus\mathbf{m}$ in this case is blocked from going to the right bulk.

$B_5 = \mathbf{1} \oplus \mathbf{e} \oplus \mathbf{m} \oplus \mathbf{f}$ can be regarded as the tensor of two commutative algebras $A_{\mathbf{e}} := \mathbf{1} \oplus \mathbf{e}$ and $A_{\mathbf{m}} := \mathbf{1} \oplus \mathbf{m}$. Here we show that the subalgebra $\mathbf{1} \oplus \mathbf{e} \cong A_{\mathbf{e}} \otimes \mathbf{1}$ of $B_5 \cong A_{\mathbf{e}} \otimes A_{\mathbf{m}}$ is the left center $Z_l(B_5)$. Indeed, the following diagram commutes:

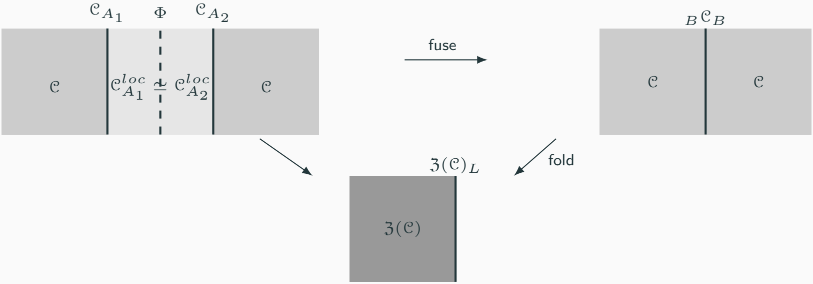
$$
\begin{array}{ccccc}
(A_{\mathbf{e}} \otimes \mathbf{1}) \otimes (A_{\mathbf{e}} \otimes A_{\mathbf{m}}) & \xrightarrow{\mathrm{id}} & (A_{\mathbf{e}} \otimes A_{\mathbf{e}}) \otimes (\mathbf{1} \otimes A_{\mathbf{m}}) & \searrow & \\
\beta_{A_{\mathbf{m}}, A_{\mathbf{e}}} \uparrow & \circlearrowleft & & & A_{\mathbf{e}} \otimes A_{\mathbf{m}} \\
(A_{\mathbf{e}} \otimes A_{\mathbf{m}}) \otimes (A_{\mathbf{e}} \otimes \mathbf{1}) & \xrightarrow[\beta_{A_{\mathbf{m}}, A_{\mathbf{e}}}]{} & (A_{\mathbf{e}} \otimes A_{\mathbf{e}}) \otimes (A_{\mathbf{m}} \otimes \mathbf{1}) & \nearrow &
\end{array}
$$

In addition, the following diagram does not commute,

$$
\begin{array}{ccccc}
(\mathbf{1} \otimes A_{\mathbf{m}}) \otimes (A_{\mathbf{e}} \otimes A_{\mathbf{m}}) & \xrightarrow{\beta_{A_{\mathbf{m}}, A_{\mathbf{e}}}} & (\mathbf{1} \otimes A_{\mathbf{e}}) \otimes (A_{\mathbf{m}} \otimes A_{\mathbf{m}}) & \searrow & \\
\beta_{A_{\mathbf{e}}, A_{\mathbf{m}}} \uparrow & \not\circlearrowleft & & & A_{\mathbf{e}} \otimes A_{\mathbf{m}} \\
(A_{\mathbf{e}} \otimes A_{\mathbf{m}}) \otimes (\mathbf{1} \otimes A_{\mathbf{m}}) & \xrightarrow[\beta_{A_{\mathbf{m}}, \mathbf{1}}]{} & (A_{\mathbf{e}} \otimes \mathbf{1}) \otimes (A_{\mathbf{m}} \otimes A_{\mathbf{m}}) & \nearrow &
\end{array}
$$

since $\beta_{A_{\mathbf{e}}, A_{\mathbf{m}}} = \mathrm{id} \oplus \mathrm{id} \oplus \mathrm{id} \oplus \beta_{\mathbf{e},\mathbf{m}} = \mathrm{id} \oplus \mathrm{id} \oplus \mathrm{id} \oplus -\mathrm{id}$ and $\beta_{A_{\mathbf{m}}, A_{\mathbf{e}}} = \mathrm{id}$. For a similar process, we have $Z_r(B_5) = \mathbf{1} \oplus \mathbf{m}$. Therefore, $Z_l(B_5) \cong \mathbf{1} \oplus \mathbf{e} \overset{2-Morita}{\sim} \mathbf{1} \oplus \mathbf{m} \cong Z_r(B_5)$.

2-Morita equivalent condensable algebras $A_1 \underset{\phi}{\overset{2-Morita}{\sim}} A_2$ in $\mathcal{C}$ can also be classified by lagrangian algebras (maximal 2d condensable algebras) L_i in $\mathcal{C} \boxtimes \overline{\mathcal{C}} \simeq \mathfrak{Z}(\mathcal{C})$. In particular, $L_i \simeq Z(B_i)$ [Kong-Runkel 09].



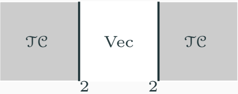
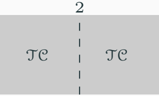
$$L \cap (\mathcal{C} \boxtimes \overline{\mathbf{1}}) = A_1 \underset{\phi}{\overset{2-Morita}{\sim}} A_2 = L \cap (\mathbf{1} \boxtimes \overline{\mathcal{C}})$$

Theorem

Given any pair of 2-Morita equivalent algebras $A_1 \underset{\phi}{\overset{2-Morita}{\sim}} A_2$ *in* $\mathcal{C}$, $\mathcal{C}_{A_1} \boxtimes_{\mathcal{C}^{loc}_{A_1}} \Phi \boxtimes_{\mathcal{C}^{loc}_{A_2}} \mathcal{C}_{A_2}$ *is equivalent to* $\mathfrak{Z}(\mathcal{C})_L$ *as monoidal* $\mathcal{C}$*-*$\mathcal{C}$*-bimodule for some lagrangian algebra* $L \in \mathfrak{Z}(\mathcal{C})$.

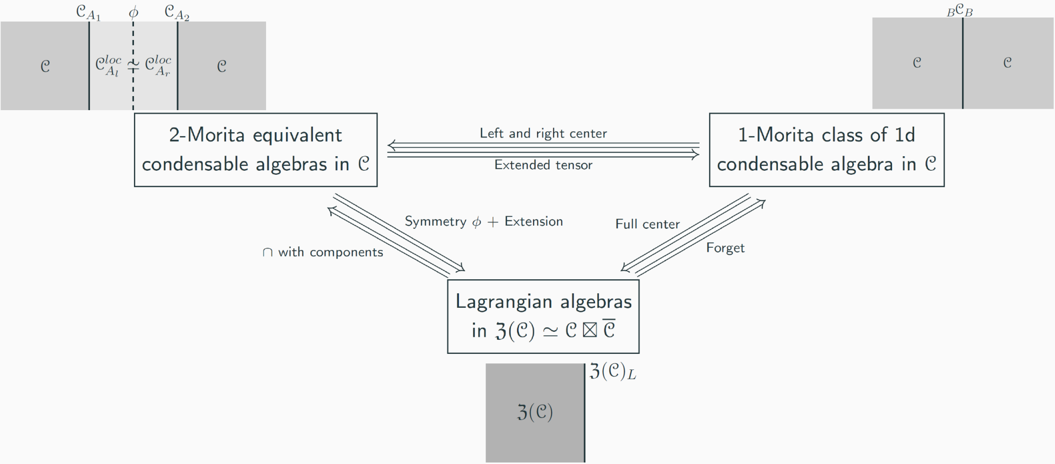
For example, by intersecting $\mathbf{1\overline{1}} \oplus \mathbf{m\overline{e}} \oplus \mathbf{e\overline{m}} \oplus \mathbf{f\overline{f}} \in \mathcal{TC} \boxtimes \overline{\mathcal{TC}}$ with $\mathcal{TC} \boxtimes \overline{\mathbf{1}}/\mathbf{1} \boxtimes \overline{\mathcal{TC}}$, we obtain $\mathbf{1} \underset{\phi}{\overset{2-Morita}{\sim}} \mathbf{1}$; similarly, starting from $\mathbf{1\overline{1}} \oplus \mathbf{e\overline{1}} \oplus \mathbf{1\overline{m}} \oplus \mathbf{e\overline{m}}$, we obtain $\mathbf{1} \oplus \mathbf{e} \overset{2-Morita}{\sim} \mathbf{1} \oplus \mathbf{m}$.

$B_i \in \mathcal{TC}$	$Z_l(B_i)/Z_r(B_i)$	Domain wall	Lagrangian algebras $L_i \in \mathcal{TC} \boxtimes \overline{\mathcal{TC}}$
$\mathbf{1}$	$\mathbf{1}/\mathbf{1}$	trivial wall	$\mathbf{1\overline{1}} \oplus \mathbf{e\overline{e}} \oplus \mathbf{m\overline{m}} \oplus \mathbf{f\overline{f}}$
$\mathbf{1} \oplus \mathbf{f}$	$\mathbf{1}/\mathbf{1}$	$\mathbf{e} - \mathbf{m}$ exchange	$\mathbf{1\overline{1}} \oplus \mathbf{m\overline{e}} \oplus \mathbf{e\overline{m}} \oplus \mathbf{f\overline{f}}$
$\mathbf{1} \oplus \mathbf{e}$	$\mathbf{1} \oplus \mathbf{e}/\mathbf{1} \oplus \mathbf{e}$	$\mathrm{Vec}_{\mathbb{Z}_2} \boxtimes \mathrm{Vec}_{\mathbb{Z}_2}$	$\mathbf{1\overline{1}} \oplus \mathbf{e\overline{1}} \oplus \mathbf{1\overline{e}} \oplus \mathbf{e\overline{e}}$
$\mathbf{1} \oplus \mathbf{m}$	$\mathbf{1} \oplus \mathbf{m}/\mathbf{1} \oplus \mathbf{m}$	$\mathrm{Rep}(\mathbb{Z}_2) \boxtimes \mathrm{Rep}(\mathbb{Z}_2)$	$\mathbf{1\overline{1}} \oplus \mathbf{m\overline{1}} \oplus \mathbf{1\overline{m}} \oplus \mathbf{m\overline{m}}$
$\mathbf{1} \oplus \mathbf{e} \oplus \mathbf{m} \oplus \mathbf{f}$	$\mathbf{1} \oplus \mathbf{e}/\mathbf{1} \oplus \mathbf{m}$	$\mathrm{Vec}_{\mathbb{Z}_2} \boxtimes \mathrm{Rep}(\mathbb{Z}_2)$	$\mathbf{1\overline{1}} \oplus \mathbf{e\overline{1}} \oplus \mathbf{1\overline{m}} \oplus \mathbf{e\overline{m}}$
$\mathbf{1} \oplus \mathbf{e} \oplus \mathbf{m} \oplus \mathbf{f}, \omega$	$\mathbf{1} \oplus \mathbf{m}/\mathbf{1} \oplus \mathbf{e}$	$\mathrm{Rep}(\mathbb{Z}_2) \boxtimes \mathrm{Vec}_{\mathbb{Z}_2}$	$\mathbf{1\overline{1}} \oplus \mathbf{m\overline{1}} \oplus \mathbf{1\overline{e}} \oplus \mathbf{m\overline{e}}$

H	F	E_2 condensable algebras in $\mathcal{TC}$	Condensed phase $\mathcal{TC}_A^{loc}$	Domain walls	Total: 6
$\mathbb{Z}_2$ $\{e\}$	$\mathbb{Z}_2$ $\{e\}$	$\mathbf{1} \oplus \mathbf{m}$ $\mathbf{1} \oplus \mathbf{e}$	Vec		non-invertible: $2 \times 2 = 4$
$\mathbb{Z}_2$	$\{e\}$	$\mathbf{1}$	$\mathcal{TC}$		invertible: 2

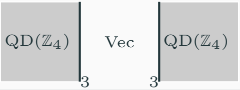
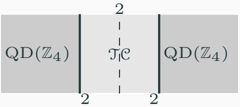
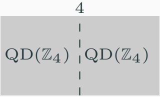
Main results [Fröhlich-Fuchs-Runkel-Schweigert06, Kong-Runkel09, Davydov10, Davydov-Nikshych-Ostrik12]:

We give a complete interplay between E_1 condensable algebras in $\mathcal{C}$, 2-Morita equivalent E_2 condensable algebras in $\mathcal{C}$, and lagrangian algebras in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$.

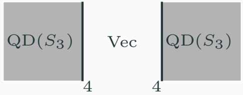
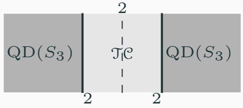
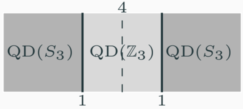
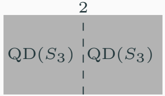


Using these equivalences, we can classify all indecomposable gapped domain walls within a topological order, especially in Kitaev quantum double models described by $\mathfrak{Z}(\mathrm{Vec}_G)$ [Kitaev03, Davydov10].

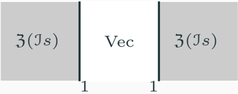
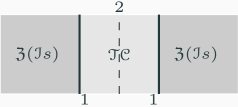
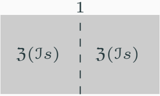
Results of $\mathfrak{Z}(\mathrm{Vec}_{\mathbb{Z}_4}) := \mathrm{QD}(\mathbb{Z}_4)$

H	F	Condensable algebras in $\mathrm{QD}(\mathbb{Z}_4)$	Condensed phase $\mathrm{QD}(\mathbb{Z}_4)_A^{loc}$	Domain walls	Total: 22
$\{e\}$ $\mathbb{Z}_2$ $\mathbb{Z}_4$	$\{e\}$ $\mathbb{Z}_2$ $\mathbb{Z}_4$	$\mathbf{1}\oplus\mathbf{e}\oplus\mathbf{e}^2\oplus\mathbf{e}^3$ $\mathbf{1}\oplus\mathbf{e}^2\oplus\mathbf{m}^2\oplus\mathbf{f}^2$ $\mathbf{1}\oplus\mathbf{m}\oplus\mathbf{m}^2\oplus\mathbf{m}^3$	Vec		non-invertible: $3\times 3 = 9$
$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbf{1}\oplus\mathbf{f}^2$	$\mathcal{DS}$		non-invertible: 1
$\mathbb{Z}_4$ $\mathbb{Z}_2$	$\mathbb{Z}_2$ $\{e\}$	$\mathbf{1}\oplus\mathbf{m}^2$ $\mathbf{1}\oplus\mathbf{e}^2$	$\mathcal{TC}$		non-invertible: $2\times 2\times 2 = 8$
$\mathbb{Z}_4$	$\{e\}$	$\mathbf{1}$	$\mathrm{QD}(\mathbb{Z}_4)$		invertible: 4

Results of $\mathfrak{Z}(\mathrm{Vec}_{S_3}) := \mathrm{QD}(S_3)$

H	F	Condensable algebras in $\mathrm{QD}(S_3)$	Condensed phase $\mathrm{QD}(S_3)_A^{loc}$	Domain walls	Total: 28
S_3 A_3 C_2 $\{e\}$	S_3 A_3 C_2 $\{e\}$	$\mathbf{A} \oplus \mathbf{F} \oplus \mathbf{D}$ $\mathbf{A} \oplus \mathbf{B} \oplus 2\mathbf{F}$ $\mathbf{A} \oplus \mathbf{C} \oplus \mathbf{D}$ $\mathbf{A} \oplus \mathbf{B} \oplus 2\mathbf{C}$	Vec		non-invertible: $4 \times 4 = 16$
S_3 C_2	A_3 $\{e\}$	$\mathbf{A} \oplus \mathbf{F}$ $\mathbf{A} \oplus \mathbf{C}$	$\mathcal{TC}$		non-invertible: $2 \times 2 \times 2 = 8$
A_3	$\{e\}$	$\mathbf{A} \oplus \mathbf{B}$	$\mathrm{QD}(\mathbb{Z}_3)$		non-invertible: $4/2 = 2$
S_3	$\{e\}$	$\mathbf{A}$	$\mathrm{QD}(S_3)$		invertible: 2

Recall that a 2d condensable algebra A in $\mathcal{C}$ may have non-trivial algebra automorphisms φ that leads to a braided autoequivalence in $\mathcal{C}_A^{loc}$. This kind of Φ_φ does not lead to a new gapped domian wall in $\mathcal{C}$ after folding.

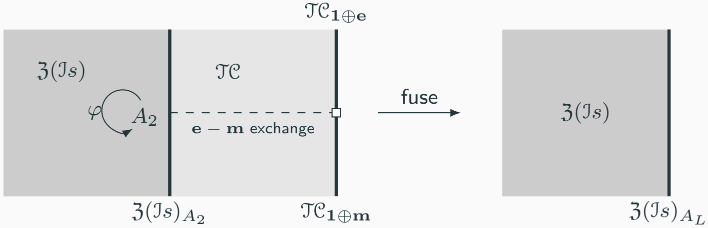
<table>
<tr><th>Condensable algebras in $\mathfrak{Z}(\mathcal{I}s)$</th><th>Condensed phase $\mathfrak{Z}(\mathcal{I}s)_A^{loc}$</th><th>Domain walls</th><th>Total: 3</th></tr>
<tr><td>$A_L := (\mathbf{1} \boxtimes \overline{\mathbf{1}}) \oplus (\psi \boxtimes \overline{\psi}) \oplus (\sigma \boxtimes \overline{\sigma})$</td><td>Vec</td><td>
</td><td>non-invertible: 1</td></tr>
<tr><td>$A_2 := (\mathbf{1} \boxtimes \overline{\mathbf{1}}) \oplus (\psi \boxtimes \overline{\psi})$</td><td>$\mathcal{TC}$</td><td>
</td><td>non-invertible: $2/2 = 1$</td></tr>
<tr><td>$A_1 := \mathbf{1} \boxtimes \overline{\mathbf{1}}$</td><td>$\mathfrak{Z}(\mathcal{I}s)$</td><td>
</td><td>invertible: 1</td></tr>
</table>

For example, double Ising topological order has only three distinguishable gapped domain walls. Although $\mathcal{TC}$ has an $\mathbf{e} - \mathbf{m}$ exchange domain wall, $\mathfrak{Z}(\mathcal{I}s)_{A_2} \boxtimes_{\mathcal{TC}} \mathfrak{Z}(\mathcal{I}s)_{A_2}$ is the unique domain wall associated to the 2-Morita class $(\mathbf{1} \boxtimes \overline{\mathbf{1}}) \oplus (\psi \boxtimes \overline{\psi})$.

This is due to the $\mathbf{e}$-$\mathbf{m}$ exchange in $\mathfrak{Z}(\mathcal{I}s)_{A_2}^{loc} \simeq \mathcal{TC}$ is induced by the non-trivial algebra automorphism φ of A_2:

$$(\mathbf{1} \boxtimes \overline{\mathbf{1}}) \oplus (\psi \boxtimes \overline{\psi}) \xrightarrow{1 \oplus -1} (\mathbf{1} \boxtimes \overline{\mathbf{1}}) \oplus (\psi \boxtimes \overline{\psi})$$

The obvious inclusion $i : A_2 \hookrightarrow A_L$ determines a 2-step condensation process. $(\mathbf{1} \boxtimes \overline{\mathbf{1}}) \oplus (\psi \boxtimes \overline{\psi})$ corresponds to $\mathbf{1} \in \mathcal{TC}$ and the component $(\sigma \boxtimes \overline{\sigma})$ corresponds to $\mathbf{e}$ (or $\mathbf{m}$, depends on the convention). So A_L becomes the lagrangian algebra $\mathbf{1} \oplus \mathbf{e}$ in $\mathcal{TC}$. After composing $i : A_2 \hookrightarrow A_L$ with φ, we obtain a new two step condensation $i' : A_2 \hookrightarrow A_L$. The component $(\mathbf{1} \boxtimes \overline{\mathbf{1}}) \oplus (\psi \boxtimes \overline{\psi})$ is invariant and still corresponds to $\mathbf{1}$, but the component $(\sigma \boxtimes \overline{\sigma})$ becomes $(\sigma \boxtimes \overline{\sigma})^{tw}$, which corresponds to $\mathbf{m}$ now. Hence, A_L has two incarnations $\mathbf{1} \oplus \mathbf{m}$ and $\mathbf{1} \oplus \mathbf{e}$ in $\mathcal{TC}$ under this condensation process.



Similarly, for $\mathrm{QD}(S_3)$,

$$\mathrm{QD}(S_3)^{loc}_{\mathbf{A}\oplus\mathbf{B}} \xrightarrow{\simeq} \mathrm{QD}(\mathbb{Z}_3)$$
$$\mathbf{A}\oplus\mathbf{B} \mapsto \mathbf{1}$$
$$\mathbf{C} \mapsto \mathbf{e} \qquad \mathbf{C}^{tw} \mapsto \mathbf{e}^2$$
$$\mathbf{F} \mapsto \mathbf{m} \qquad \mathbf{F}^{tw} \mapsto \mathbf{m}^2$$

- The $\mathbf{e}-\mathbf{m}$ exchange domain wall in $\mathrm{QD}(\mathbb{Z}_3)$ is induced from the $\mathbf{C}-\mathbf{F}$ exchange domain wall in $\mathrm{QD}(S_3)$.
- $1-2$ exchange domain wall is induced from the algebra automorphism $\varphi : \mathbf{A}\oplus\mathbf{B} \xrightarrow{1\oplus -1} \mathbf{A}\oplus\mathbf{B}$.

Note that $\mathrm{QD}(\mathbb{Z}_3)$ has two gapped boundaries $\mathrm{Rep}(\mathbb{Z}_3)$ and $\mathrm{Vec}_{\mathbb{Z}_3}$ obtained by condensing $\mathbf{1}\oplus\mathbf{m}\oplus\mathbf{m}^2$ and $\mathbf{1}\oplus\mathbf{e}\oplus\mathbf{e}^2$, in which $1-2$ exchange acts trivially on these two lagrangian algebras. Extending these two lagrangian algebras back to $\mathrm{QD}(S_3)$, we obtain $\mathbf{A}\oplus\mathbf{B}\oplus 2\mathbf{F}$ and $\mathbf{A}\oplus\mathbf{B}\oplus 2\mathbf{C}$ which generate $\mathrm{Vec}^{\mathbf{F}}_{S_3}$ and Vec_{S_3} respectively. φ in this case does not generate more gapped boundaries for the condensed phase.