Equivariant K-homology of affine Grassmannian

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§.0. Motivation

- What is Schubert calculus ?
- Prototye
- Main Theorem

§.1. Setting

- Root data of GL_n
- 0 - Grassmann elements

§.2. Main tools

- Demazure operator
- level 0 affine K-nil Hecke algebras.

§.3. Double K-k-Schur function

§0. What is Schubert calculus?

$\boxed{\text{Classical}}$

$\mathbb{P}^3$

The Problem of Four Lines

by H. Schubert

$\rightsquigarrow$

$\boxed{\text{Modern}}$

Schubert class

$$H^*(Gr(2,\mathbb{C}^4)) = \bigoplus_{\lambda \subset \boxplus} \mathbb{Z}\,\sigma_\lambda$$

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu \subset \boxplus} c_{\lambda\mu}^{\nu}\, \sigma_\nu$$

The structure constants of a product.

$$Gr(2,\mathbb{C}^4) := \{ V \subset \mathbb{C}^4 \mid \dim V = 2 \}$$

vector sp.

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§0. Prototype

$$Gr(d, \mathbb{C}^n) = \{ V < \mathbb{C}^n \mid \dim V = d \}$$

vector sp.

$$\frac{\mathbb{Z}[z_1, \dots, z_d]^{\mathfrak{S}_d}}{\langle S_\lambda(z) \mid \lambda_1 > n-d \rangle} \xrightarrow{\sim} H^*(Gr(d, \mathbb{C}^n)) = \bigoplus \mathbb{Z}\sigma_\lambda$$

$\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d$,

$n-d \geqq \lambda_1 \geqq \cdots \geqq \lambda_d \geqq 0$

$$\overline{S_\lambda(z)} \longmapsto \sigma_\lambda$$

$$S_\lambda(z) := \frac{\det(z_j^{\lambda_i + d - i})}{\det(z_j^{d-i})} \in \mathbb{Z}[z_1, \dots, z_d]^{\mathfrak{S}_d} \quad : \quad \text{Schur poly.}$$

irreducible character of rep. of GL_d

Example $(d=2)$

$S_{\square}(z_1, z_2) = z_1 + z_2$, $S_{\square\square}(z_1, z_2) = z_1^2 + z_1 z_2 + z_2^2$

$S_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(z_1, z_2) = z_1 z_2$, $S_{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}}(z_1, z_2) = z_1^2 z_2^2$

§0. Main theorem (1/2)

$$Gr(d, \mathbb{C}^n) \cong GL_n / P_d \ , \quad P_d := \left\{ \begin{array}{cc} & d \\ d & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array} \right\} \subset GL_n$$

parabolic sub group.

$$\rightsquigarrow$$

$$Gr_{GL_n} = GL_n(\mathbb{C}((t))) / GL_n(\mathbb{C}[[t]])$$

As the set of $\mathbb{C}$-valued pt.

The Object of Our Interest

K-Peterson isom.

$$K_*^T(Gr_{GL_n}) := \bigoplus_{w \in \widetilde{W}^0_{GL_n}} R(T)[\mathcal{O}_w] \left(\xrightarrow{\sim} QK_T(GL_n/B) \right)$$

Quantum K-theory of flag var.

$$T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subset GL_n \ , \quad R(T) \cong \mathbb{Z}[e^{\pm a_1}, \dots, e^{\pm a_n}]$$

maximal torus

Representation ring of T

Main Theorem [Ikeda – Shimozono – Y.]

$$y = (y_1, y_2, \cdots), \quad b = (b_1, \cdots, b_n)$$

$$(b_i = 1 - e^{-a_i})$$

$$K_*^T(Gr_{GL_n}) \cong \bigoplus_{w \in \tilde{W}^\circ_{GL_n}} R(T)\, \tilde{g}_w^{(k)}(y|b)$$

Symmetric function

$$\begin{array}{ccc} K_*^T(Gr_{GL_n}) & & \bigoplus R(T)\, \tilde{g}_w^{(k)}(y|b) \\ \Psi & & \Psi \\ [\mathcal{O}_{X_w}] & \longmapsto & \tilde{g}_w^{(k)}(y|b) \end{array}$$

$k = n-1$

($X_w \subset Gr_{GL_n}$, $\mathcal{O}_{X_w}$: structure sheaf of X_w.)

Schubert var

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§1 Root data of GL_n

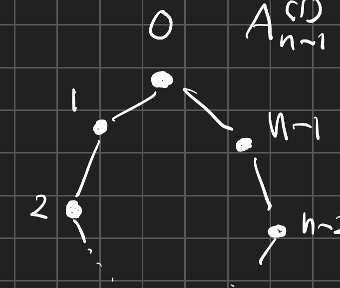
A_{n-1}

$$G = GL_n(\mathbb{C}) \supset T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} = (\mathbb{C}^\times)^n$$

$$X := \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{C}^\times) = \bigoplus_{i \in I} \mathbb{Z} a_i \supset Q = \bigoplus_{i \in I} \mathbb{Z} \overset{a_i - a_{i+1}}{\overset{\shortparallel}{\alpha_i}} \supset \Phi := \{ a_i - a_j \mid i \neq j \}$$

$$X^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}^\times, T) = \bigoplus_{i \in I} \mathbb{Z} \varepsilon_i \supset Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \underset{\varepsilon_i - \varepsilon_{i+1}}{\underset{\shortparallel}{\alpha_i^\vee}} \supset \Phi^\vee := \{ \varepsilon_i - \varepsilon_j \mid i \neq j \}$$

$A_{n-1}^{(1)}$



$$W_G := \langle s_1, \cdots, s_{n-1} \rangle \cong \mathfrak{S}_n$$

$\cap$

$$\widehat{W}_G := Q^\vee \rtimes W_G = \langle \underset{t_{\theta^\vee} s_\theta}{\underset{\shortparallel}{s_0}}, s_1, \cdots, s_{n-1} \rangle \qquad s_\theta = (1, n)$$

$\cap$

$$\widetilde{W}_G := X^\vee \rtimes W_G = \widehat{W}_G \rtimes \underset{\pi_1(G) \cong \mathbb{Z}}{\left(X^\vee / Q^\vee \right)} = \langle s_0, s_1, \cdots, s_{n-1} ; \underset{t_{\varepsilon_1} s_1 \cdots s_{n-1}}{\underset{\shortparallel}{\tau}} \rangle$$

§1. 0-Grassmann elements

$$\widetilde{W}_G^0 := \{ w \in \widetilde{W}_G \mid \ell(ws_i) = \ell(w) + 1 \quad (i \in I) \}$$

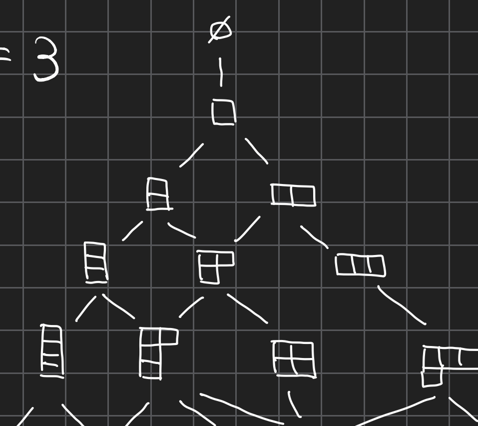
$$\wr\!\wr$$

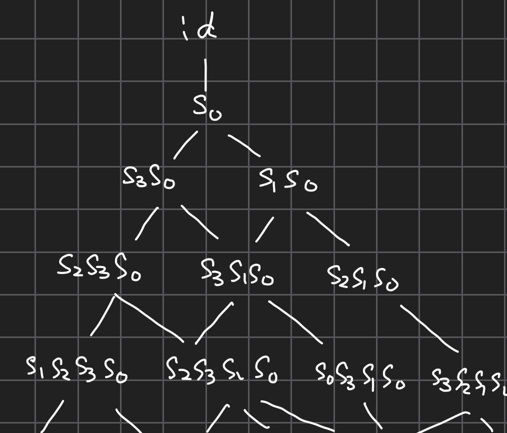
$$\mathcal{P}^{n-1 = k} := \{ \lambda = (\lambda_1, \lambda_2 \cdots) \mid \underset{\overset{\shortparallel}{k}}{n-1} \geq \lambda_1 \geq \lambda_2 \geq \cdots \}$$

k-bounded partition.

Example $\quad n = 4, \; k = 3$

0	1	2	3	0	1	2	3	⋯
3	0	1	2	3	0	1	2	⋯
2	3	0	1	2	3	0	1	⋯
1	2	3	0	1	2	3	0	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	





$$\lambda = (3, 2, 1) \longmapsto w_\lambda = s_2 \cdot s_0 s_3 \cdot s_2 s_1 s_0$$

§1. Demazure operators

$$\widetilde{W}_G \curvearrowright R(T), \quad t_\lambda w(e^\mu) = e^{w(\mu)}, \quad t_\lambda \in X^\vee, w \in W_G$$

level 0 action

$$R(T)^\Delta := R(T)[[(1-e^\alpha)^{-1} \mid \alpha \in \Phi]]$$

$$\mathbb{K}_G^\Delta := R(T)^\Delta \otimes_{\mathbb{Z}} \mathbb{Z}[\widetilde{W}_G] \qquad (f \otimes w)(g \otimes v) = f w(g) \otimes wv$$

$$(f, g \in R(T)^\Delta, \ w, v \in W_G)$$

$$T_i := \frac{s_i - 1}{1 - e^{\alpha_i}}, \quad D_i := T_i + 1 \in \mathbb{K}_G^\Delta \qquad (i \in \widetilde{I} := I \cup \{0\})$$

Demazure operators $(\alpha_0 = -\theta = a_n - a_1)$

§.1. level 0 affine K-nil Hecke algebras

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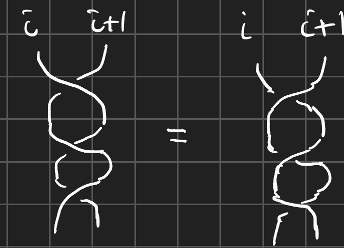
$$T_i^2 = -T_i \qquad \& \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$
$$D_i^2 = D_i \qquad\qquad D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}$$



(braid relation)

$$\rightsquigarrow \quad \widetilde{W}_G \ni w = \tau^{\ell} s_{i_1} \cdots s_{i_k} \ \text{: reduced} \quad \Longrightarrow \quad \begin{cases} T_w = \tau^{\ell} T_{i_1} \cdots T_{i_k} \\ D_w = \tau^{\ell} D_{i_1} \cdots D_{i_k} \end{cases}$$

$$\mathbb{K}_G := \bigoplus_{w \in \widetilde{W}_G} R(T) T_w \qquad \text{affine } K\text{-nil-Hecke ring.}$$

§2. Double K-k-Schur functions (1/3)

- $k = n-1$
- $b_i = 1 - e^{-a_i} \in R(T)$
- Λ : The ring of symmetric function $y = (y_1, y_2, \cdots)$
- $\hat{\Lambda}$: The completed ring of Λ

$$\mathbb{K}_G \curvearrowright \hat{\Lambda}_{R(T)} := R(T) \otimes_{\mathbb{Z}} \hat{\Lambda} \ni D_w(1) =: \tilde{g}_w^{(k)}(y|b)$$

$\mathbb{K}_G \supset W_G$, $\mathbb{K}_G \supset X^\vee$

$(w \in \tilde{W}_G^0)$

"Double K-k-Schur functions"

$$t_{\varepsilon_i} \leftrightarrow \Omega(b_i|y) \times = \prod_{j=1}^{\infty} \frac{1}{1-b_i y} \times$$

- $w(b_i) = b_{w(i)}$
- $w(y) = y$

§2. Double K-k-Schur functions (2/3)

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Thm [Ikeda – Shimozono – Y.]

$$K_*^T(Gr_G) \cong \hat{\Lambda}_{(n)} := \mathbb{K}_G \cdot 1 = \sum_{w \in \tilde{W}_G^0} R(T)\, \tilde{g}_w^{(k)}(y|b)$$

$$[\mathcal{O}_{X_w}] \longmapsto \tilde{g}_w^{(k)}(y|b)$$

Example

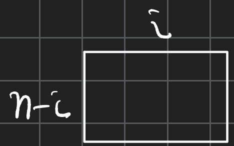
$$\tilde{g}_{s_0}^{(k)}(y|b) = D_0(1) = \frac{\overset{s_0}{\overset{\parallel}{t_{\theta^\vee} s_\theta}}(1) - 1}{1 - e^{\alpha_0}} + 1 = \frac{t_{\theta^\vee}(1) - 1}{1 - e^{a_n - a_1}} + 1$$

$$= \frac{1}{1 - e^{a_n - a_1}} \left(\frac{\Omega(b_1|y)}{\Omega(b_n|y)} - 1 \right) + 1$$

Schur poly.

$$= e^{-a_n} \sum_{p \geq 1, q \geq 0} (-1)^q b_n^q b_1^{p-i}\, s_{(p,1^q)}(y) + 1$$

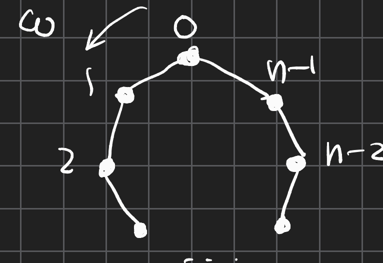
§2. Double K-k-Schur functions (3/3)

$R_i =$ 

$\in \mathcal{P}^k$

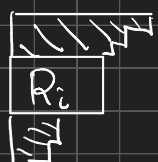
rectangle class

$\omega(b_i) = b_{i+1}$



Th'm [Ikeda - Iwao - Naito - Y., IST]

$$\tilde{g}^{(k)}_{\lambda \cup R_i}(y|b) = \tilde{g}^{(k)}_{\lambda}(y|\omega^i(b))\,\tilde{g}^{(k)}_{R_i}(y|b)$$



$\lambda \cup R_i$