Supersymmetric vertex algebras

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1. Vertex algebras and supersymmetric vertex algebras

Introduction and some observation on supersymmetric vertex algebras, partly based on

Takumi Iwane, S.Y., "SUSY structure of chiral de Rham complex from the factorization structure",

arXiv:2409.04220.

1. Definitions of (supersymmetric) vertex algebras [5 pages]
   1.1. Vertex algebras
   1.2. Supersymmetric vertex algebras
2. Invariants of (supersymmetric) vertex algebras
3. Factorization structures
4. Open problems

A vertex algebra (VA) is an algebraic structure introduced by Borcherds[1] to formulate algebraic aspects of two-dimensional conformal field theory.

It is defined to be a $\mathbb{C}$-linear space V equipped with vertex operators

$$V \to (\operatorname{End} V)[\![z^{\pm 1}]\!], \quad a \mapsto a(z) = \sum_{j\in\mathbb{Z}} z^{-j-1} a_{(j)}$$

and $|0\rangle \in V$, satisfying for any $a, b \in V$ that

- $a(z)b \in V(\!(z)\!) := \{\sum_{n=k}^{\infty} v_n z^n \mid \exists k \in \mathbb{Z},\ v_n \in V\}$ (quantum fields),
- $(|0\rangle)(z) = \mathrm{id}_V,\ a(z)\,|0\rangle = a + O(z)$ (vacuum),
- $T \in \operatorname{End} V,\ a \mapsto a_{(-2)}\,|0\rangle$ satisfies $[T, a(z)] = (Ta)(z) = \partial_z a(z)$ (translation),
- $\exists N_{a,b} \in \mathbb{Z}_{\geq 0}$ s.t. $(z-w)^{N_{a,b}}[a(z), b(w)] = 0$ (locality).

V being a $\mathbb{C}$-linear superalgebra, we have vertex superalgebras (VSA).

[1] R. Borcherds, "Vertex algebras, Kac-Moody algebras, and the Monster", Proc. Nat. Acad. Sci., **83** (1986), no. 10, 3068–71.

- V: VA, $V \to (\mathrm{End}\, V)[\![z^{\pm 1}]\!]$, $a \mapsto a(z) = \sum_{j\in\mathbb{Z}} z^{-j-1} a_{(j)}$.
- λ-bracket encodes (j)-operations: $[a_\lambda b] := \sum_{j\geq 0} \frac{1}{j!}\lambda^j a_{(j)} b \in V[\lambda]$.
- $\omega \in V$ is conformal of central charge $c \in \mathbb{C}$
  if $[\omega_\lambda \omega] = (T + 2\lambda)\omega + \frac{c}{12}\lambda^3$. (Virasoro relation)

Example (universal affine VA). $\mathfrak{g}$: simple Lie algebra.
$\widehat{\mathfrak{g}} := \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}K$: affine Kac-Moody Lie algebra of $\mathfrak{g}$:

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K, \quad [K, \widehat{\mathfrak{g}}] = 0.$$

$V^k(\mathfrak{g}) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} \mathbb{C}_k$: vacuum module of level $k \in \mathbb{C}$.

$V^k(\mathfrak{g})$ has a unique VA structure with $|0\rangle = 1 \otimes 1$ and vertex operators

$$xt^{-1}|0\rangle \mapsto \sum_{j\in\mathbb{Z}} z^{-j-1}(xt^j) \quad (x \in \mathfrak{g}).$$

If $k \neq -h^\vee$, Sugawara vector $\omega_{\mathrm{Sug}} := \frac{1}{2(k+h^\vee)} \sum_a (J_a t^{-1})(J^a t^{-1})|0\rangle$
is conformal with $c = \frac{k \dim \mathfrak{g}}{k+h^\vee}$.

($\{J^a\}_a \subset \mathfrak{g}$: basis, $\{J_a\}_a$: dual basis w.r.t. invariant form on $\mathfrak{g}$)

A supersymmetric vertex algebra[2] (SUSY VA) is an extension of VA encoding two-dimensional supersymmetric CFTs.
$Z = (z, \zeta)$: supervariable. $Z^{j|J} := z^j \zeta^J$ $(j \in \mathbb{Z},\ J = 0, 1)$.

An $N = 1$ SUSY VA is a $\mathbb{C}$-linear superspace V with even $|0\rangle \in V$ and

$$V \to (\operatorname{End} V)[\![Z^{\pm 1}]\!], \quad a \mapsto a(Z) = \sum_{j \in \mathbb{Z},\, J=0,1} Z^{-j-1|1-J} a_{(j|J)},$$

satisfying for any $a, b \in V$ that

- $a(Z)b \in V(\!(Z)\!) := \{\sum_{j=k,\, J}^{\infty} v_{j|J} Z^{j|J} \mid \exists k \in \mathbb{Z},\ v_{j|J} \in V\}$,
- $(|0\rangle)(Z) = \mathrm{id}_V$, $a(Z)\,|0\rangle = a + O(Z)$, (vacuum)
- $S \in \operatorname{End} V$, $a \mapsto a_{(-1|1)}\,|0\rangle$ satisfies $(Sa)(Z) = (\partial_\zeta + \zeta\partial_z)a(Z)$ and $[S, a(Z)] = (\partial_\zeta - \zeta\partial_z)a(Z)$, (odd translation)
- $\exists N_{a,b} \in \mathbb{Z}_{\geq 0}$ s.t. $(z - w)^{N_{a,b}}[a(Z), b(W)] = 0$. (locality)

2 R. Heluani, V. Kac, "Supersymmetric Vertex Algebras", Comm. Math. Phys., **271** (2007), 103–178.

- V: $N = 1$ SUSY VA, $V \ni a \mapsto a(Z) = \sum_{j \in \mathbb{Z},\, J=0,1} Z^{-j-1|1-J} a_{(j|J)}$.
- $T := S^2$ is the even translation: $(Ta)(Z) = [T, a(Z)] = \partial_z a(Z)$.
- Λ-bracket encodes $(j|J)$-operations: $[a_\Lambda b] := \sum_{j \geq 0,\, J} \pm \frac{1}{j!} \Lambda^{j|J} a_{(j|J)} b$.
  $\pm := (-1)^{|J|N + \binom{|J|+1}{2}}$, $\Lambda^{j|J} := \lambda^j \chi^J$, λ: even, χ: odd, $[\chi, \chi] = 2\lambda$.

Motivational example ($N = 1$ superconformal VA).
VSA V having even ω (Virasoro) and odd ν (Neveu-Schwarz) with

$$[\omega_\lambda \omega] = (T + 2\lambda)\omega + \frac{c}{12}\lambda^3, \quad [\omega_\lambda \nu] = (T + \frac{3}{2}\lambda)\nu, \quad [\nu_\lambda \nu] = 2\omega + \frac{c}{3}\lambda^2.$$

(Recall $[a_\lambda b] := \sum_{n \geq 0} \frac{1}{n!} \lambda^n a_{(n)} b$ for $a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} a_{(n)}$.)

It is an $N = 1$ SUSY VA by
$S := \nu_{-\frac{1}{2}}$, $\nu(z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} z^{-n-\frac{3}{2}} \nu_n$ and $a \mapsto a(Z) := a(z) + \zeta(Sa)(z)$.

In particular, $\omega = 2S\nu$, $\nu(Z) = \nu(z) + 2\zeta\omega(z)$ and

$$[\nu_\Lambda \nu] = \chi[\nu_\lambda \nu] + 2[\omega_\lambda \nu] = (2T + \chi S + 3\lambda)\nu + \frac{c}{3}\lambda^2 \chi.$$

Extension to N-supersymmetry: $Z = (z, \zeta^1, \dots, \zeta^N)$.
$Z^{j|J} = z^j \zeta^J := z^j \zeta^{j_1} \cdots \zeta^{j_r}$ $(j \in \mathbb{Z},\ J = \{j_1, \dots, j_r\} \subset [N] := \{1, \dots, N\})$.

An (*N*-)SUSY VA is a $\mathbb{C}$-linear superspace V with even $|0\rangle \in V$ and

$$V \to (\mathrm{End}\, V)[\![Z^{\pm 1}]\!],\ \ a \mapsto a(Z) = \sum_{j \in \mathbb{Z},\, J \subset [N]} Z^{-j-1|[N] \setminus J} a_{(j|J)},$$

satisfying similar axioms for $N = 1$, except for

- $S_i \in \mathrm{End}\, V$, $a \mapsto a_{(-1|e_i)} |0\rangle$ $\quad (e_i := \{i\} \subset [N]$ for $i = 1, \dots, N)$
  is an odd translation:
  $(S_i a)(Z) = (\partial_{\zeta^i} + \zeta^i \partial_z) a(Z)$ and $[S, a(Z)] = (\partial_{\zeta^i} - \zeta^i \partial_z) a(Z)$.

$T := S_1^2 = \cdots = S_N^2$ is an even translation: $(Ta)(Z) = [T, a(Z)] = \partial_z a(Z)$.

Motivational examples are $N = 2$ and $N = 4$ superconformal VAs.

2. Invariants of (supersymmetric) vertex algebras

1. Definitions of (supersymmetric) vertex algebras
2. Invariants of (supersymmetric) vertex algebras [6 pages]
   2.1. Zhu's C_2-Poisson and associative algebras
   2.2. Supersymmetric C_2-Poisson algebra
   2.3. Supersymmetric Zhu algebra ?
3. Factorization structures
4. Open problems

Invariants of VAs: R_V and A_V [3]

- C_2-Poisson algebra R_V describes Poisson structure in classical limit:
  $R_V := V/C_2(V), \quad C_2(V) := \langle a_{(-p)}b \mid a, b \in V,\ p \geq 2\rangle_{\mathrm{lin}}.$
  $\overline{a} \cdot \overline{b} := \overline{a_{(-1)}b}, \quad \{\overline{a}, \overline{b}\} := \overline{a_{(0)}b}.$
- Zhu (associative) algebra A_V describes the representation theory:
  (when V is a VOA: $\omega_{(1)}$ is semisimple, $V = \bigoplus_\Delta V_\Delta$)
  $A_V := V/(V \circ V), \ \ V \circ V := \langle \sum_{n\geq 0} \binom{\Delta_a}{n} a_{(n-2)}b \mid a, b \in V\rangle_{\mathrm{lin}}.$
  $[a] * [b] := [\sum_{n\geq 0} \binom{\Delta_a}{n} a_{(n-1)}b].$
  simple A_V-mods $\overset{1:1}{\longleftrightarrow}$ simple h.wt. V-mods.

[3] Y. Zhu, "Modular Invariance of Characters of Vertex Operator Algebras", J. Amer. Math. Soc., **9** (1996), no. 1, 237–302.

Example 1 (universal affine VA $V^k(\mathfrak{g})$). $\mathfrak{g}$: simple Lie algebra, $k \in \mathbb{C}$.
$V^k(\mathfrak{g}) \coloneqq U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k$, $xt^{-1}|0\rangle \mapsto \sum_{n\in\mathbb{Z}} (xt^n) z^{-n-1}$ $(x \in \mathfrak{g})$.

- C_2-Poisson algebra: $R_{V^k(\mathfrak{g})} \cong$ Kostant-Kirillov Poisson algebra of $\mathfrak{g}$:
$$R_{V^k(\mathfrak{g})} \cong \mathbb{C}[\mathfrak{g}^*], \quad \overline{(x_1 t^{-1}) \cdots (x_r t^{-1})|0\rangle} \leftrightarrow x_1 \cdots x_r \quad (x_i \in \mathfrak{g}),$$
$$\{x_i, x_j\}_{\mathbb{C}[\mathfrak{g}^*]} \coloneqq [x_i, x_j]_{\mathfrak{g}}.$$
- Zhu algebra[4]: $A_{V^k(\mathfrak{g})} \cong U(\mathfrak{g})$, $[x] \leftrightarrow x$ $(x \in \mathfrak{g})$.
For $k \neq -h^\vee$, the conformal vector ω_{Sug} gives $[\omega_{\mathrm{Sug}}] \in Z(A_{V^k(\mathfrak{g})})$.

[4] I. B. Frenkel, Y. Zhu, "Vertex operator algebras associated to representations of affine and Virasoro algebras", Duke Math. J., **66** (1992), 123–168.

Example 2 (W-algebras[5]).

$\mathfrak{g}$: simple Lie algebra, $k \in \mathbb{C}$, $f \in \mathfrak{g}$: (regular) nilpotent element.

$\rightsquigarrow$ quantum Drinfeld-Sokolov (BRST) reduction $H^\bullet_{DS,f}(?)$ of $V^k(\mathfrak{g})$:

$$\mathcal{W}^k(\mathfrak{g}, f) := H^0_{DS,f}(V^k(\mathfrak{g})).$$

C_2-Poisson algebra[6]: $R_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathbb{C}[S_f]$, $S_f := f + \mathfrak{g}^e$: Slodowy slice.

Zhu algebra[7] $A_{\mathcal{W}^k(\mathfrak{g},f)} \cong U(\mathfrak{g}, f)$: finite W-algebra.

Sub-example: $\mathcal{W}^k(\mathfrak{sl}_2, f = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]) \cong Vir_c$, Virasoro VA, $c = 1 - \frac{6(k+1)^2}{k+2}$.

$\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$, $[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}$.

$M_c := U(\mathcal{L}) \otimes_{U(\mathcal{L}_{\geq 0})} \mathbb{C}_c$, $\mathcal{L}$-module, $c \in \mathbb{C}$.

$Vir_c := M_c / U(\mathcal{L})L_{-1}(1 \otimes 1)$, $\omega = L_{-2}\left|0\right\rangle \mapsto \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$.

C_2- and Zhu algebras: $R_{Vir_c} \cong A_{Vir_c} \cong \mathbb{C}[x]$, $x := [\omega]$.

[5] B. Feigin, E. Frenkel, "Quantization of the Drinfeld-Sokolov reduction", Phys. Lett. B, **246** (1990), 75–81.

[6] A. De Sole, V. G. Kac, "Finite vs affine W-algebras", Jpn. J. Math., **1** (2006), 137–261.

[7] T. Arakawa, "Associated varieties of modules over Kac-Moody algebras and C_2-cofiniteness of W-algebras", IMRN (2015), 11605–666.

2.2. Supersymmetric C_2-Poisson algebra [1/2]

- C_2-Poisson algebra: $R_V = V/C_2(V)$, $\overline{a}\cdot\overline{b} := \overline{a_{(-1)}b}$, $\{\overline{a},\overline{b}\} := \overline{a_{(0)}b}$.
  R_V is a part of the graded Poisson VA assoc. to the Li filtration[8]:
  $F^pV := \langle a^1_{(-1-j_1)}\cdots a^r_{(-1-j_r)}b \mid r\geq 0,\, a^i, b\in V,\, j_i\geq 0,\, \sum_i j_i \geq p\rangle_{\mathrm{lin}}$,
  $V = F^0V \supset F^1V \supset F^2V \supset \cdots$,
  $R_V = F^0V/F^1V \subset \mathrm{gr}_F\, V := \bigoplus_{p\geq 0} F^pV/F^{p+1}V$.
- We have a natural SUSY analogue[9]: for an N-SUSY VA V,
  $F^pV := \langle a^1_{(-1-j_1|J_1)}\cdots a^r_{(-1-j_r|J_r)}b \mid \text{—},\, J_i\subset [N]\rangle_{\mathrm{lin}}$,
  $R_V := F^0V/F^1V \subset \mathrm{gr}_F\, V := \bigoplus_{p\geq 0} F^pV/F^{p+1}V$.
  $\mathrm{gr}_F\, V$ is a graded SUSY Poisson VA.
  R_V is a Poisson superalgebra with $\overline{a}\cdot\overline{b} := \overline{a_{(-1|[N])}b}$,
  $\{\overline{a},\overline{b}\} := \overline{a_{(0|[N])}b}$, and odd derivations $\overline{S_i}$.

[8] H. Li, "Abelianizing Vertex Algebras", Comm. Math. Phys., **259** (2005), 391–411.
[9] S.Y., "Li filtrations of SUSY vertex algebras", Lett. Math. Phys., **112** (2022), Article no. 103, 77pp.

Example 1 (Neveu-Schwarz SUSY VA).
$H :=$ $\mathbb{C}$-superalgebra generated by odd S. $T := S^2$.
$V := H[\nu] = \mathbb{C}[S^n\nu \mid n \geq 0]$, free H-module with odd ν,
is an $N = 1$ SUSY VA with Λ-bracket $[\nu_\Lambda \nu] = (2T + \chi S + 3\lambda)\nu + \frac{c}{3}\lambda^2$.

C_2-Poisson superalgebra: $R_V \cong \mathbb{C}[\overline{\nu}, \overline{S\nu}] = \mathbb{C}[\overline{\nu}, \overline{\omega}]$ $(\omega := \frac{1}{2}S\nu)$,
with non-trivial Poisson bracket $\{\overline{\nu}, \overline{\nu}\} = 2\overline{\omega}$ and odd derivation $\overline{S}$.

Example 2 (bc-$\beta\gamma$ system).
$V := H[B^1, \ldots, B^n, \Psi_1, \ldots, \Psi_n]$: free H-mod. with even B^i and odd Ψ_i.
V is an $N = 1$ SUSY VA with non-trivial Λ-bracket $[B^i{}_\Lambda \Psi_j] = \delta^i_j$ and
Neveu-Schwarz $\nu := \sum_{i=1}^n ((SB^i)_{(-1|1)}(S\Psi_i) + (TB^i)_{(-1|1)}\Psi_i)$, $c = 3n$.

$B^i(Z) = \beta^i(z) + \zeta b^i(z)$, $\Psi_i(Z) = c_i(z) + \zeta\gamma_i(z) \rightsquigarrow bc$- and $\beta\gamma$-systems.

C_2-Poisson: $R_V \cong \mathbb{C}[\overline{B^i}, \overline{SB^i}, \overline{\Psi_i}, \overline{S\Psi_i}]$, $\{\overline{B^i}, \overline{S\Psi_j}\} = \{\overline{SB^i}, \overline{\Psi_j}\} = \delta^i_j$.

At present, there seems no SUSY analogue of Zhu's associative algebra.

An ad-hoc analogue is:

$$a_{[h]}b := \operatorname{sres}_Z(a(Z)h(Z)b)dZ = a(Z)h(Z)b\big|_{Z^{-1|[N]}} \text{ for } h(Z) \in \mathbb{C}[\![Z]\!],$$
$$A_V := V/V_{[\partial_z f]}V, \quad [a] * [b] := [a_{[f]}b]$$
$$\text{with } f(Z) = f(z, \zeta^1, \ldots, \zeta^N) := ce^{cz}\zeta^{[N]}/(e^{cz}-1) \quad (c \in \mathbb{C},\ c \neq 0).$$

This $(A_V, *)$ is associative, and independent of c up to isomorphism.
For $N = 0$, it recovers the non-SUSY case[10].
In particular, $[a_{[f]}b] = [\sum_{n\geq 0} \binom{\Delta_a}{n} a_{(n-1)}b]$ for VOA.

But it doesn't seem to care about the SUSY structure...

[10] Y.-Z. Huang, "Differential equations, duality and modular invariance", Comm. Contemp. Math., **7** (5) (2005), 649–706.

3. Factorization structure

1. Definitions of (supersymmetric) vertex algebras
2. Invariants of (supersymmetric) vertex algebras
3. Factorization structure [6 pages]
   3.1. Chiral de Rham complex and formal loop space
   3.2. Factorization and VA structure
   3.3. Supersymmetry via superloop space
4. Open problems

X: a smooth algebraic variety X over $\mathbb{C}$.
Ω_X^{ch}: chiral de Rham complex[11], a sheaf of dg vertex algebras on X.

Sections on open $U \subset X$ with coordinate $x^1, \ldots, x^n$ are bc-$\beta\gamma$ system:

$$\beta^i_{(-1)} \longleftrightarrow x^i, \ \ \gamma_{i,(-1)} \longleftrightarrow \partial_{x^i}, \ \ b^i_{(-1)} \longleftrightarrow dx^i, \ \ c_{i,(-1)} \longleftrightarrow \partial_{dx^i}.$$

The dg structure of Ω_X^{ch} is:

$d_{\mathrm{ch}} := \sum_{i,n} \gamma_{i,(n)} b^i_{(-n-1)}$, differential,

$F := \sum_{i,n} : b^i_{(n-1)} c_{i,(-n)} :$, grading, $\Omega_X^{\mathrm{ch}} = \bigoplus_{p \in \mathbb{Z}} \Omega_X^{\mathrm{ch},p}$.

Ω_X^{ch} contains the de Rham complex $\Omega_X^* = \bigoplus_{p \geq 0} \Omega_X^p$:

$$\Omega_X^* \cong \mathbb{C}[\beta^i_{(-1)}] \otimes \textstyle\bigwedge^*[b^i_{(-1)}] \hookrightarrow \Omega_X^{\mathrm{ch}}, \ \ d_{\mathrm{dR}} = \sum_i \gamma_{i,(0)} b^i_{(-1)}.$$

Non-trivial point: bc-$\beta\gamma$ systems glue to give a sheaf Ω_X^{ch}.
Known is obstruction theory for the existence of VA sheaves[12]

[11] F. Malikov, V. Schechtman, A. Vaintrob, "Chiral de Rham complex", Comm. Math. Phys., **204** (1999), 439–73.

[12] V. Gorbunov, F. Malikov, V. Schechtman, "Gerbes of chiral differential operators II", Inv. math., **155** (2004), 605–680.
P. Bressler, "The first Pontryagin class", Compos. Math., **143** (2007), 1127–1163.

3.1. Chiral de Rham complex and formal loop space

Kapranov and Vasserot[13] gave a geometric construction of the chiral de Rham complex Ω_X^{ch} as the de Rham complex of formal loop space $\mathcal{L}X$.

$\mathcal{L}X$: "space of maps from punctured formal disc to X",
an ind-scheme representing $S \mapsto \mathrm{Hom}_{\mathrm{Lsp}}((\underline{S}, \mathcal{O}_S((t))^{\surd}), X)$.
$\mathcal{O}_S((t))^{\surd} \subset \mathcal{O}_S((t))$: consisting of $\sum_{n \gg -\infty}^{\infty} a_n t^n$, a_n is nilpotent for $n < 0$.

$\mathcal{J}X$: jet scheme of X, "space of maps from formal disc to X",
representing $S \mapsto \mathrm{Hom}_{\mathrm{Lsp}}((\underline{S}, \mathcal{O}_S[\![t]\!]), X)$.
We have natural morphisms $X \xleftarrow{p} \mathcal{J}X \xhookrightarrow{i} \mathcal{L}X$.

Using the description $\mathcal{L}X = \varprojlim_n \varinjlim_{\varepsilon} \mathcal{L}_n^{\varepsilon} X$, one can define $\mathrm{DR}(\mathcal{M})$ for a right $\mathcal{D}$-module $\mathcal{M}$ on $\mathcal{L}X$. Covering X by affine open U, we have a sheaf $\mathcal{DR}(\mathcal{M})$: $U \mapsto \mathrm{DR}(\mathcal{M}|_{\mathcal{L}U})$ on X. Applying it to $\mathcal{M} := i_* p^* \omega_X$, we have a sheaf of complexes $\mathcal{DR}(i_* p^* \omega_X)$. Then $\Omega_X^{\mathrm{ch}} \cong \mathcal{DR}(i_* p^* \omega_X)$.

[13] M. Kapranov, E. Vasserot, "Vertex algebras and the formal loop space", Publ. Math. IHES **100** (2004), 209–269.

Description of $\mathrm{DR}(i_* p^* \omega_{\mathbb{A}^d})$ in terms of bc-$\beta\gamma$ system:

$Y^n_m := \operatorname{Spec} \mathbb{C}[\beta^i_k \mid -n \le k < m,\ i = 1, \dots, d]$, $\quad \beta^i_0 \leftrightarrow (x^i \text{ on } \mathbb{A}^d)$.

$\mathcal{L}^n_m := \operatorname{Spf} \mathbb{C}[\beta^i_k \mid 0 \le k < m][\![\beta^i_k \mid -n \le k < 0]\!]$: compl. of Y^n_m along Y^0_m.

Then $\mathcal{L}\mathbb{A}^d = \varprojlim_m \varinjlim_n \mathcal{L}^n_m$ and $\mathrm{DR}(i_* p^* \omega_{\mathbb{A}^d}) = \varinjlim_{m,n} \mathrm{DR}(\omega^n_m)[-dm]$,

$\omega^n_m := (i_{m,n})_* \omega_{Y^0_m}$, $\ i_{m,n} \colon Y^0_m \hookrightarrow Y^n_m$: closed embedding.

$$\mathcal{DR}(\omega^n_m)\colon\ \omega^n_m \otimes \bigwedge^{d(m+n)} \Theta \to \cdots \to \omega^n_m \otimes \bigwedge^{dm} \Theta \to \cdots \to \omega^n_m \otimes \Theta \to \omega^n_m.$$

$\Theta := \Theta_{Y^n_m}$, tangent sheaf of Y^n_m.

$\Gamma(\omega^n_m) \cong D^n_m / (\beta^i_k, \gamma^i_l \mid -n \le k < 0 \le l < m,\ i) D^n_m$, $\qquad \beta^i_0 \leftrightarrow x^i$, $\gamma^i_0 \leftrightarrow \partial_{x^i}$

$D^n_m := \mathbb{C}\langle \beta^i_k, \gamma^i_k \mid -n \le k < m,\ i \rangle / ([\beta^i_k, \beta^j_l], [\gamma^i_k, \gamma^j_l], [\beta^i_k, \gamma^j_l] - \delta^{i,j}\delta_{k+l,0})$.

$\Gamma(\bigwedge^* \Theta_{Y^n_m}) \cong C^n_m / (b^i_k \mid -n \le k < m,\ i) C^n_m$, $\qquad b^i_0 \leftrightarrow dx^i$, $c^i_0 \leftrightarrow \partial_{dx^i}$

$C^n_m := \mathbb{C}\langle b^i_k, c^i_k \mid -n \le k < m,\ i \rangle / ([b^i_k, b^j_l]_+, [c^i_k, c^j_l]_+, [b^i_k, c^j_l]_+ - \delta^{i,j}\delta_{k+l,0})$.

$$\mathrm{DR}(\omega^n_m)[-dm] : (CD^n_m)^{d(m+n)} \to \cdots \to (CD^n_m)^{dm} \to \cdots \to (CD^n_m)^0.$$

$CD^n_m := D^n_m \otimes C^n_m$, $\ (CD^n_m)^p \subset CD^n_m$: part of fermionic charge p.

$\rightsquigarrow \mathrm{DR}(i_* p^* \omega_{\mathbb{A}^d}) \cong CD = d$-dimensionoal bc-$\beta\gamma$ system $= \Gamma(\Omega^{\mathrm{ch}}_{\mathbb{A}^d})$.

The VA structure of $\mathcal{DR}(i_* p^* \omega_X)$ ($\cong \Omega_X^{\mathrm{ch}}$) comes from factorization structure of global loop spaces $(\mathcal{L}_{C^I} X)_I$.

X: smooth algebraic variety. I: non-empty finite set.

C: smooth algebraic curve, where "the coordinate z" lives.

$\mathcal{L}_{C^I} X$: "space of maps from I-tuples of punctured discs on C to X.

$$S \mapsto \{(f, \rho) \mid f \in \mathrm{Hom}_{\mathrm{Sch}}(S, C^I),\ \rho \in \mathrm{Hom}_{\mathrm{Lsp}}((\underline{\Gamma(f)}, \mathcal{K}^{\surd}_{\Gamma(f)}), X)\}.$$

$\Gamma(f) \subset S \times C$: union of graphs of $f = (f_i)_{i \in I}$.

For $n = 1, 2, \ldots,$ $Y_n := \mathcal{L}_{C^n} X$ $\ (I = \{1, \ldots, n\})$.

$\Delta \colon C \hookrightarrow C^2$: diagonal embedding $\rightsquigarrow$ natural isom. $\nu \colon \Delta^* Y_2 \xrightarrow{\sim} Y_1$.

$\because$ $(\Delta^* Y_2)(S) \ni ((f_1, f_2), \rho)$, $(f_1, f_2) : S \to C^2$ lying in the image of Δ,
so f_i factor through $\exists!\ f : S \to C$, and $\Gamma((f_1, f_2)) = \Gamma(f)$.

$j \colon U \hookrightarrow C^2$: complement of $\Delta(C)$ $\rightsquigarrow$ $\kappa \colon j^*(Y_1 \times Y_1) \xrightarrow{\sim} j^* Y_2$.

$\because$ $(j^* Y_2)(S) \ni ((f_1, f_2), \rho)$, $(f_1, f_2) : S \to U \subset C^2$, and $\Gamma((f_1, f_2)) = \Gamma(f_1) \sqcup \Gamma(f_2)$.

For any $p\colon J \twoheadrightarrow I$, we have $\nu_p\colon \Delta_p^* Y_J \xrightarrow{\sim} Y_I$, $\kappa_p\colon j_p^*(\prod_i Y_{p^{-1}(i)}) \to j_p^* Y_J$.

$\Delta_p\colon C^I \hookrightarrow C^J$: partial diagonal, $j_p\colon U_p \hookrightarrow C^J$: complement.

These $(\nu_p, \kappa_p)_p$ satisfy certain compatibility: factorization structure.

The sheaf of complexes $\mathcal{CDR}(X) := \mathcal{DR}(i_* p^* \omega_X)$ ($\cong \Omega_X^{\mathrm{ch}}$) on X

$\rightsquigarrow$ a sheaf of complexes $\mathcal{CDR}_n$ of left $\mathcal{D}_{C^n}$-modules on $X \times C^n$ such that

(the fiber of $\mathcal{CDR}_1$ at $c \in C$) $\cong \mathcal{CDR}(X)$,

$\nu\colon \Delta^* \mathcal{CDR}_2 \cong \mathcal{CDR}_1$, $\kappa\colon j^*(\mathcal{CDR}_{X,C} \boxtimes \mathcal{CDR}_{X,C}) \xrightarrow{\sim} j^* \mathcal{CDR}_{X,C^2}$.

$\rightsquigarrow$ left $\mathcal{D}$-module morphism

$$\mu\colon j_* j^*(\mathcal{CDR}_1 \boxtimes \mathcal{CDR}_1) \overset{\kappa}{\cong} j_* j^* \mathcal{O}_{C^2} \otimes \mathcal{CDR}_2 \xrightarrow{\text{canon.}} \Delta_* \mathcal{O}_C \otimes \mathcal{CDR}_2 \overset{\nu}{\cong} \Delta_* \mathcal{CDR}_1.$$

The morphism μ gives the VA structure of $\mathcal{CDR}(X) \cong \Omega_X^{\mathrm{ch}}$.

We can explain[14] the $N = 1$ SUSY VA structure of Ω_X^{ch} in terms of the factorization structure of the superloop space $(\mathcal{LS}_{(C,\mathcal{S})^I} X)_I$.

$(C, \mathcal{S})$: superconformal curve. C: smooth supercurve of dimension $1|1$.
$\mathcal{S} \subset \Theta_C$: odd line subbundle s.t. $[\cdot,\cdot]$ mod $\mathcal{S}$: $\bigwedge^2 \mathcal{S} \xrightarrow{\sim} \Theta_C/\mathcal{S}$. Locally on $(C, \mathcal{S})$, exists the superconformal coordinate $Z = (z, \zeta)$ s.t. $\mathcal{S}$ is generated by $\not{\partial}_Z := \partial_\zeta + \zeta \partial_z$.

$\mathcal{LS}X$: formal superloop space, $S \mapsto \mathrm{Hom}_{\mathrm{Lsp}}((\underline{S}, \mathcal{O}_S(\!(t)\!)^{\surd}[\eta]), X)$, η: odd.
$\mathcal{LS}_{(C,\mathcal{S})^I} X$: global superloop space, having factorization structure.

$\Delta^s \subset C^2$: superdiagonal defined by $z_1 - z_2 - \zeta_1\zeta_2 = 0$.
$j^s \colon U^s \hookrightarrow C^2$: complement.

$\rightsquigarrow$ a $\mathcal{D}$-module morphism $\mu^s \colon j^s_* j^{s*}(\mathcal{CDR}_1^{\boxtimes 2}) \to \Delta^s_* \mathcal{CDR}_1$ on C^2, which yields the $N = 1$ SUSY structure of $\mathcal{CDR}(X) \cong \Omega_X^{\mathrm{ch}}$.

[14] T. Iwane, S.Y., "SUSY structure of chiral de Rham complex from the factorization structure", arXiv:2409.04220.

4. Open problems of supersymmetric vertex algebras

- SUSY analogue of Zhu's associative algebra A_V (see Page 14)
- Reduction of SUSY VAs (Page 11)
- Obstruction theory of SUSY chiral differential operators (Page 16)
- Geometric explanation (e.g. via factorization) of higher SUSY structure of chiral de Rham complex[15]
- SUSY analogue of chiral homology[16]

$\vdots$

Thank you.

[15] D. Ben-Zvi, R. Heluani, M. Szczesny, "Supersymmetry of the chiral de Rham complex", Compos. Math., **144** (2008), 503–521.
[16] A. Beilinson, V. Drinfeld, "Chiral Algebras", AMS Colloquium Publ., **51**, Amer. Math. Soc., Providence, RI, 2004.