The 2-character theory of finite 2-groups

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Tannaka-Krein duality for finite 2-groups. Mo Huang and Zhi-Hao Zhang. arXiv: 2305.18151

The 2-character theory for finite 2-groups. Mo Huang, Hao Xu and Zhi-Hao Zhang. arXiv: 2404.01162

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Introduction

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- The category $\mathrm{Rep}(G)$ of finite-dimensional representations of G is finite semisimple (Maschke's theorem). Here we work over $\Bbbk = \mathbb{C}$. Equivalently, the group algebra $\Bbbk[G]$ is semisimple.
- For $V \in \mathrm{Rep}(G)$, its character $\chi_V \colon G \to \Bbbk$ defined by $\chi_V(g) := \mathrm{tr}_V(\rho(g))$ is conjugation invariant: $\chi_V(h) = \chi_V(ghg^{-1})$ for all $g, h \in G$. So the characters are class functions.

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- There is an inner product $\langle \cdot, \cdot \rangle$ on the space $\mathrm{Fun}_c(G, \Bbbk)$ of class functions and we have the orthogonality of characters:

$$\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W), \quad \forall V, W \in \mathrm{Rep}(G).$$

  In particular, if $V, W \in \mathrm{Rep}(G)$ are irreducible representations, the inner product $\langle \chi_V, \chi_W \rangle$ is 1 if $V \simeq W$, otherwise it is 0.

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We are going to generalize these results in the 2-representation theory of finite 2-groups.

Definition

A **2-group** is a monoidal category $(\mathcal{G}, \otimes, \mathbb{1})$ in which every morphism is an isomorphism and each object x is invertible, i.e., there exists an object y such that $x \otimes y \simeq \mathbb{1} \simeq y \otimes x$.

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Let $\mathcal{G}$ be a 2-group. Its **first homotopy group** $\pi_1(\mathcal{G})$ is defined to be the group of isomorphism classes of objects in $\mathcal{G}$, and its **second homotopy group** $\pi_2(\mathcal{G})$ is defined to be the automorphism group $\mathrm{Hom}_{\mathcal{G}}(\mathbb{1}, \mathbb{1})$ of the tensor unit of $\mathcal{G}$.

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The 2-group structure of $\mathcal{G}$ is determined by its homotopy groups $\pi_1(\mathcal{G}), \pi_2(\mathcal{G})$ and the 3-cohomology class $[\alpha] \in H^3(\pi_1(\mathcal{G}); \pi_2(\mathcal{G}))$ (the Postnikov class or the k-invariant), where the 3-cocycle $\alpha \in Z^3(\pi_1(\mathcal{G}); \pi_2(\mathcal{G}))$ is the associator of $\mathcal{G}$, and $\pi_1(\mathcal{G})$ acts on the abelian group $\pi_2(\mathcal{G})$ by conjugation.

2-representation and 2-group 2-algebra $\mathrm{Vec}_{\mathcal{G}}$

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There is a '2-group 2-algebra' $\mathrm{Vec}_{\mathcal{G}}$ defined in the following two steps:

- linearizing the hom spaces: $\mathcal{G} \rightsquigarrow \Bbbk_* \mathcal{G}$;
- taking the **Karoubi completion**: $\Bbbk_* \mathcal{G} \rightsquigarrow \mathsf{Kar}(\Bbbk_* \mathcal{G}) =: \mathrm{Vec}_{\mathcal{G}}$.

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By the universal property of Karoubi completion, finite semisimple 2-representations of $\mathcal{G}$ are equivalent to finite semisimple modules over $\mathrm{Vec}_{\mathcal{G}}$. Therefore, it categorifies the group algebra of a finite group.

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When $\mathcal{G}$ is finite, $\mathrm{Vec}_{\mathcal{G}}$ is a **multi-fusion category** (i.e., finite semisimple rigid monoidal $\Bbbk$-linear category, which is a categorification of semisimple algebra). Therefore, $2\mathrm{Rep}(\mathcal{G}) \simeq \mathsf{LMod}_{\mathrm{Vec}_{\mathcal{G}}}(2\mathrm{Vec})$ is a finite semisimple 2-category.

Theorem [Douglas-Reutter: 1812.11933]

A linear 2-category D is finite semisimple iff there exists a multi-fusion category $\mathcal{C}$ such that D is equivalent to the 2-category $\mathsf{LMod}_{\mathcal{C}}(2\mathrm{Vec})$ of finite semisimple left $\mathcal{C}$-modules.

The symmetric fusion 2-category $2\mathrm{Rep}(\mathcal{G})$

Moreover, for two finite semisimple 2-representations $\mathcal{V}, \mathcal{W} \in 2\mathrm{Rep}(\mathcal{G})$, their Deligne tensor product $\mathcal{V} \boxtimes \mathcal{W}$ admits a diagonal $\mathcal{G}$-action:

$$g \odot (v \boxtimes w) := (g \odot v) \boxtimes (g \odot w), \quad g \in \mathcal{G},\ v \in \mathcal{V},\ w \in \mathcal{W}.$$

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$$g \odot^* v := g^{-1} \odot v, \quad g \in \mathcal{G},\, v \in \mathcal{V}.$$

The evaluation 1-morphism is $\mathrm{Hom}\colon \mathcal{V}^{\mathrm{op}} \boxtimes \mathcal{V} \to \mathrm{Vec}$ and the coevaluation 1-morphism is

$$\begin{aligned} \mathrm{Vec} &\to \mathcal{V} \boxtimes \mathcal{V}^{\mathrm{op}} \\ \Bbbk &\mapsto \bigoplus_{x \in \mathrm{Irr}(\mathcal{V})} x \boxtimes x \end{aligned}$$

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Therefore, $2\mathrm{Rep}(\mathcal{G})$ is a symmetric fusion 2-category (i.e., finite semisimple rigid symmetric monoidal 2-category such that $\mathbb{1}$ is simple) in the sense of Douglas and Reutter.

2-character

Recall that for a group G the character of a representation $(V, \rho) \in \mathrm{Rep}(G)$ is the function $G \to \Bbbk$ defined by $g \mapsto \mathrm{tr}(\rho(g))$. Now we want to categorify this notion.

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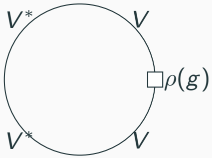
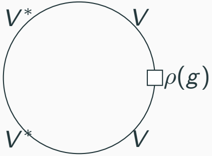
First we note that taking traces in Vec can be generalized in any rigid symmetric monoidal category (see the following graph calculus):



2-character



Then the rigidity of $2\mathrm{Vec}$ provides a categorified notion of trace. For a finite 2-group $\mathcal{G}$ and $\mathcal{V} \in 2\mathrm{Rep}(\mathcal{G})$, we define the **2-character** $\chi_{\mathcal{V}}$ by

$$\chi_{\mathcal{V}}(g) := \left(\mathrm{Vec} \xrightarrow{\text{coev}} \mathcal{V}^{\mathrm{op}} \boxtimes \mathcal{V} \xrightarrow{1 \boxtimes (g \odot -)} \mathcal{V}^{\mathrm{op}} \boxtimes \mathcal{V} \xrightarrow{\text{ev}} \mathrm{Vec}\right), \quad \forall g \in \mathcal{G}.$$

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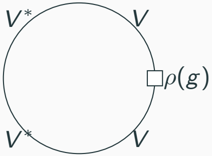

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Since $\mathrm{End}_{\Bbbk}(\mathrm{Vec}) \simeq \mathrm{Vec}$, we can identify $\chi_{\mathcal{V}}(g)$ with a vector space. Thus $\chi_{\mathcal{V}} \in \mathrm{Fun}(\mathcal{G}, \mathrm{Vec})$.

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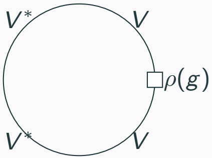

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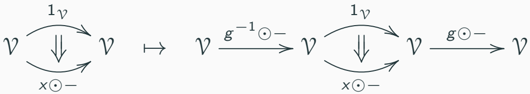
$$\chi_{\mathcal{V}}(g) \simeq \bigoplus_{v \in \mathrm{Irr}(\mathcal{V})} \mathrm{Hom}_{\mathcal{V}}(v, g \odot v) \simeq \mathrm{Nat}(1_{\mathcal{V}}, g \odot -).$$

This coincides with the "categorical trace" defined by Ganter and Kapranov [Ganter-Kapranov: math/0602510].

Conjugation invariance of 2-characters

The 2-character $\chi_{\mathcal{V}}$ has the following conjugation invariance: for any $g, x \in \mathcal{G}$ there is a canonical isomorphism of vector spaces

$$\psi_{g,x} \colon \chi_{\mathcal{V}}(x) = \mathrm{Nat}(1_{\mathcal{V}}, x \odot -) \to \mathrm{Nat}(1_{\mathcal{V}}, gxg^{-1} \odot -) = \chi_{\mathcal{V}}(gxg^{-1})$$



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Moreover, these isomorphisms render the following diagram commutes:

$$\begin{array}{ccc}
\chi_{\mathcal{V}}(x) & \xrightarrow{\psi_{g \otimes h, x}} & \chi_{\mathcal{V}}(((g \otimes h) \otimes x) \otimes (g \otimes h)^{-1}) \\
{\scriptstyle \psi_{h,x}} \downarrow & & \downarrow {\scriptstyle \simeq} \\
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These isomorphisms provide $\chi_\mathcal{V}$ a structure of a **homotopy $\mathcal{G}$-fixed point**, where the $\mathcal{G}$-action is induced by the conjugation. Thus $\chi_\mathcal{V}$ is promoted to an object in the **equivariantization** (homotopy fixed point) $\mathrm{Fun}(\mathcal{G}, \mathrm{Vec})^\mathcal{G}$. We also call an object $F \in \mathrm{Fun}(\mathcal{G}, \mathrm{Vec})^\mathcal{G}$ a **class functor**.

Conjugation invariance of 2-characters

There is an equivalence

$$\Phi\colon \mathrm{Vec}_{\mathcal{G}} \simeq \mathrm{Vec}_{\mathcal{G}}^{\mathrm{op}} \simeq \mathsf{Fun}_{\Bbbk}(\mathrm{Vec}_{\mathcal{G}}, \mathrm{Vec}) \simeq \mathsf{Fun}(\mathcal{G}, \mathrm{Vec}),$$

where the first equivalence is induced by the functor $\mathcal{G} \simeq \mathcal{G}^{\mathrm{op}}$ that sends each morphism to its inverse, the second equivalence is the Yoneda embedding, and the last one is the universal property of Karoubi completion. This categorifies the Fourier transform for finite abelian groups.

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Furthermore, Φ is an equivalence of $\mathcal{G}$-modules, where the $\mathcal{G}$-actions on both sides are induced by conjugation. Thus we obtain an equivalence between their equivariantizations:

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Remark: The monoidal structure on $\mathsf{Fun}(\mathcal{G}, \mathrm{Vec})$ transferred from $\mathrm{Vec}_{\mathcal{G}}$ is the Day convolution:

$$(F \circledast G)(g) := \int^{x \in \mathcal{G}} F(g \otimes x^{-1}) \otimes G(x), \quad F, G \in \mathrm{Vec}_{\mathcal{G}},\ g \in \mathcal{G}.$$

Algebra structure on the 2-character

Main Theorem [Huang-Xu-Z.: 2404.01162]

Let $\mathcal{G}$ be a finite 2-group and $\mathcal{V} \in 2\mathrm{Rep}(\mathcal{G})$ be an irreducible 2-representation. There is a canonical isomorphism $\Phi(Z(\mathcal{V})) \simeq \chi_{\mathcal{V}^{\mathrm{op}}}$ in $\mathrm{Fun}(\mathcal{G}, \mathrm{Vec})^{\mathcal{G}}$. Hence the irreducible 2-character $\chi_{\mathcal{V}}$ admits a structure of a Lagrangian algebra in $\mathrm{Fun}(\mathcal{G}, \mathrm{Vec})^{\mathcal{G}}$, inherited from the full center $Z(\mathcal{V}^{\mathrm{op}})$.

- A **Lagrangian algebra** [Davydov-Müger-Nikshych-Ostrik: 1009.2117] in a braided fusion category $\mathcal{B}$ is a commutative connected separable algebra that is maximal in the sense that the category $\mathrm{Mod}_A^{\mathrm{loc}}(\mathcal{B})$ of local A-modules is equivalent to Vec.
- $Z(\mathcal{V}) \in \mathfrak{Z}_1(\mathrm{Vec}_{\mathcal{G}})$ is the **full center** of $\mathcal{V}$ [Davydov: 0908.1250], which is a Lagrangian algebra [Kong-Runkel: 0807.3356].

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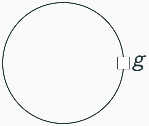
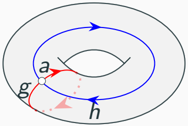
Moreover, the isomorphism classes of Lagrangian algebras in $\mathfrak{Z}_1(\mathrm{Vec}_{\mathcal{G}})$ are one-to-one corresponding to the equivalence classes of indecomposable $\mathrm{Vec}_{\mathcal{G}}$-modules [Davydov-Müger-Nikshych-Ostrik: 1009.2117], i.e., irreducible 2-representations of $\mathcal{G}$. Thus the 2-representations are completely determined by their 2-characters.

A finite semisimple 2-representation $\mathcal{V}$ is fully dualizable. By the cobordism hypothesis [Baez-Dolan: q-alg/9503002, Lurie: 0905.0465], there is a 2D $\mathcal{G}$-TQFT $Z_{\mathcal{V}} \colon \mathrm{Cob}_2^{\mathcal{G}} \to 2\mathrm{Vec}$ which maps a single point of $\mathcal{V}$.

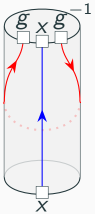
Here are some cobordisms and their images under $Z_{\mathcal{V}}$:



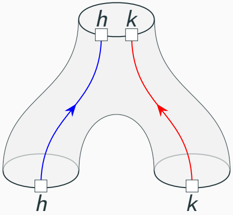
2-character $\chi_{\mathcal{V}}(g)$



joint 2-character



conjugation invariance



multiplication

Inner product on $\mathrm{Fun}(\mathcal{G}, \mathrm{Vec})^{\mathcal{G}}$

We define the **innre product** of two class functors $F, G \in \mathrm{Fun}(\mathcal{G}, \mathrm{Vec})^{\mathcal{G}}$ to be hom space

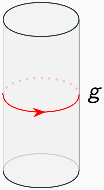
$$\langle F, G \rangle := \mathrm{Hom}_{\mathrm{Fun}(\mathcal{G},\mathrm{Vec})^{\mathcal{G}}}(\mathbb{1}, F \circledast G).$$

This inner product $\langle F, G \rangle$ is isomorphic to the $\pi_1(\mathcal{G})$-fixed point of the following space:

$$(F \circledast G)(e) = \int^{g \in \mathcal{G}} F(g^{-1}) \otimes G(g),$$

where the $\pi_1(\mathcal{G})$-action on $(F \circledast G)(e)$ is induced by the $\mathcal{G}$-equivariant structure on $F \circledast G$.

Indeed, for every class functor $F \in \mathrm{Fun}(\mathcal{G}, \mathrm{Vec})^{\mathcal{G}}$ there is a canonical $\pi_1(\mathcal{G})$-action on $F(e)$. For $F = \chi_{\mathcal{V}}$ this action is given by the following cobordism:



If $\mathcal{X}$ is a finite semisimple category, we define its **dimension** to be the $\Bbbk$-algebra

$$\dim(\mathcal{X}) := \mathrm{End}(\mathcal{X}, \mathcal{X}) \simeq \bigoplus_{x \in \mathrm{Irr}(\mathcal{X})} \mathrm{Hom}(x, x).$$

In particular, for $\mathcal{V} \in 2\mathrm{Rep}(\mathcal{G})$ we have $\dim(\mathcal{V}) \simeq \chi_{\mathcal{V}}(e)$.

Theorem [Huang-Xu-Z.: 2404.01162]

Let $\mathcal{G}$ be a finite 2-group and $\mathcal{V}, \mathcal{W} \in 2\mathrm{Rep}(\mathcal{G})$. There is a canonical isomorphism of $\Bbbk$-algebras $\langle \chi_{\mathcal{V}}, \chi_{\mathcal{W}} \rangle \simeq \dim(\mathrm{Fun}_{\mathcal{G}}(\mathcal{V}, \mathcal{W}))$.

Thanks for listening!