# Unitarity of Minimal Principal Series Representations

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September 2025

# Mathematical formulation of quantum mechanics

- Around 100 years ago, a mathematical formulation of quantum mechanics began to take shape.
- More precisely, a quantum mechanical system is described by a Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$ (inner product space + completeness) called **phase space**.
- Physical properties of the system can be described by **self-adjoint (or Hermitian) operators** $A : \mathcal{H} \to \mathcal{H}$.
- The eigenvalues $\lambda$ of $A$ ($\lambda \in \mathbb{R}$ since $A$ is self-adjoint) give all possible measurements of its corresponding physical property.

**Example**

The (time independent) phase space corresponding to the hydrogen atom is

$$\mathcal{H} = L^2(\mathbb{R}^3).$$

Consider the **Schrödinger operator** $\widehat{H} : \mathcal{H} \to \mathcal{H}$. Eigenvalues of $\widehat{H}$ give all energy levels of the atom. Write $\mathcal{H}_\lambda$ as the $\lambda$-eigenspace of $\widehat{H}$ ($\lambda$-**eigenstates**).

# Unitary representations of Lie groups

## Definition

Let $G$ be a real reductive Lie group (e.g. $SO(3)$, $SL(2,\mathbb{R})$, $U(p,q)$, $Sp(2n,\mathbb{R})$, $\dots$). A **unitary representation** of $G$ is a Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$ equipped with a continuous group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ satisfying:

$$\langle \pi(g)v, \pi(g)w\rangle = \langle v, w\rangle, \qquad (v, w \in \mathcal{H}).$$

- As an example, let $G = SO(3)$ be the Lie group of $3 \times 3$ real orthogonal matrices, and $\mathcal{H} = L^2(\mathbb{R}^3)$ be the hydrogen atom model.
- For all $g \in G$ and $f \in \mathcal{H}$, let
$$[\pi(g)f](\mathbf{x}) := f(g^{-1}\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^3.$$
This gives a unitary representation of $SO(3)$ on $\mathcal{H}$.
- Indeed, $\pi(g)$ is related to the **angular momentum operator** of the hydrogen atom model.

# Hydrogen atom and Lie algebra representation

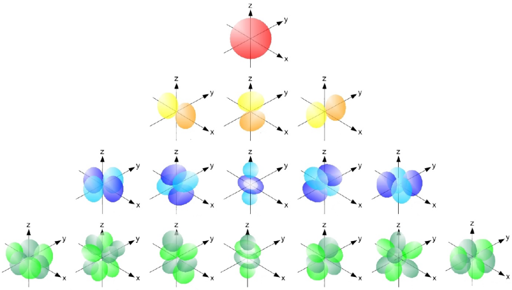
- Moreover, $\pi(g)$ commutes with the Schrodinger operator $\widehat{H}$, i.e.

$$\pi(g) \circ \widehat{H} = \widehat{H} \circ \pi(g).$$

- It turns out that $\mathcal{H} = L^2(\mathbb{R}^3)$ can be decomposed into:

$$L^2(\mathbb{R}^3) = \widehat{\bigoplus}_i \mathcal{H}_{\lambda_i}, \text{ with } \lambda_1 < \lambda_2 < \dots \text{ (the energy levels)},$$

where each $\mathcal{H}_{\lambda_i}$ is a $(2n-1)$-dimensional representation of $SO(3)$:



# The unitary dual

We are interested in the following:

**Question**

*Classify all unitary representations of a real reductive group $G$.*

- It is well known that if $(\pi, \mathcal{H})$ is unitary, then it can be decomposed as a (completed) direct sum of irreducible representations ($(\pi, \mathcal{H})$ is **irreducible** if the only $G$-invariant closed subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$ only).
- Let $K$ be a maximal compact subgroup of $G$, and $\mathfrak{g} = Lie(G) \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of the Lie algebra of $G$.
- By an old work of Harish-Chandra, the (functional analytic) problem of classifying irreducible unitary representations can be reduced to the (algebraic) problem of classifying all unitarizable, irreducible $(\mathfrak{g}, K)$-modules.
- We denote the latter as $\widehat{G}$, **the unitary dual of** $G$.

# Unitary representations of Lie groups

- **The big question: Give a real reductive Lie group $G$, classify its unitary dual $\widehat{G}$.**
- Here is a timeline on some **full** classification of $\widehat{G}$:

| | | | | | |
|---|---|---|---|---|---|
| Compact connected $G$ (Cartan-Weyl, 1920-30s) | $\to$ | $GL(2,\mathbb{R})$ (Bargmann, 1947) | $\to$ | Complex rank 2 groups (Duflo, 1979) | $\to$ |
| Real rank 1 groups (Baldoni-Silva-Barbasch, 1983) | $\to$ | $GL(n,F)$ ($F=\mathbb{R},\mathbb{C},\mathbb{H}$) (Vogan, 1986) | $\to$ | $SO(n,\mathbb{C})$ & $Sp(2n,\mathbb{C})$ (Barbasch, 1989) | $\to$ |
| Universal cover of $GL(n,\mathbb{R})$ (Huang, 1990) | $\to$ | $\widetilde{G_2}(\mathbb{R})$ (Vogan, 1994) | $\to$ | $Spin(n,\mathbb{C})$, $U(n,2)$ (W.-Zhang) | |

# The atlas program

- One of the many remarkable achievements between 1990's - 2020's in the unitary dual problem is the implementation of the atlas computer program:


Figure: The character formulas of irreducible representations for real split $E_8$ were completely computed in 2007, which gained a lot of media attention.

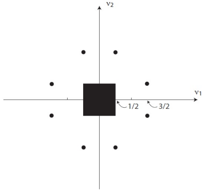
- In 2016, the is_unitary command was introduced in atlas, so that one can check whether a given irreducible representation is unitary or not.
- However, it does not give a self-included description of $\widehat{G}$. As an analogy, one can write a computer program is_prime to check whether an integer $n$ is prime or not. Yet it does not give a classification of all prime numbers.

# Spherical Unitary Dual of Split Classical Groups

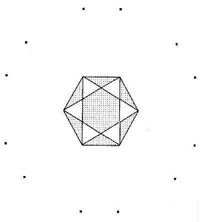
- Let $G$ be a *split* reductive group (e.g. $G = O(n,n), O(n,n+1), Sp(2n,\mathbb{R})$).
- Recall $K$ is a maximal compact subgroup of $G$ (e.g. $K = O(n) \times O(n)$, $O(n) \times O(n+1)$, $U(n)$).
- We are interested in classifying all irreducible, unitary representations of $G$ *with a $K$-fixed vector*, i.e. the **spherical unitary dual** $\widehat{G}_{sph}$.
- The classification can be reduced to the following question:

> For $\nu \in \mathbb{R}^n$, when's the **spherical Langlands quotient** $J(\nu) \in \widehat{G}_{sph}$?

- $\widehat{G}_{sph}$ is known by Barbasch-Ciubotaru, Vogan or `atlas`. For $G = O(3,2)$ and $G_2(\mathbb{R})$. the $\nu \in \mathbb{R}^2$ such that $J(\nu) \in \widehat{G}_{sph}$ are:




Unitary dual of $G = O(3,2)$



Unitary dual of $G = G_2(\mathbb{R})$

# Unitary Dual with Fine Lowest $K$-types

- We can generalize the classification of $\widehat{G}_{sph}$ - find all irreducible, unitary representations of $G$ containing a *fine* $K$-type $\delta_{p,q}$ $(p+q=n)$:
  - $G = O(n,n)$: $\delta_{p,q} = \wedge^p \mathbb{C}^n \otimes \mathbb{C}$.
  - $G = O(n,n+1)$: $\delta_{p,q} = \wedge^p \mathbb{C}^n \otimes \mathbb{C}$.
  - $G = Sp(2n,\mathbb{R})$: $\delta_{p,q} = \wedge^p \mathbb{C}^{2n}$.
- If $p = 0$, then $\delta_{0,n}$ is the trivial $K$-type and we are back to the spherical case in last slide.
- For $\nu_p \in \mathbb{R}^p$, $\nu_q \in \mathbb{R}^q$, one wants to know:

| When is the Langlands quotient $J(\delta_{p,q}; \nu_p, \nu_q)$ unitary? |
|---|

- Indeed, it's related to the *endoscopic subgroup* $G_1 \times G_2$ of $G$:

**Theorem (Barbasch-Ciubotaru-Pantano)**

$$\textit{Let } (G_1, G_2) = \begin{cases} (O(p,p), O(q,q)) & \textit{if } G = O(n,n) \\ (O(p,p+1), O(q,q+1)) & \textit{if } G = O(n,n+1) \\ (O(p,p), Sp(2q,\mathbb{R})) & \textit{if } G = Sp(2n,\mathbb{R}) \end{cases}$$

*Then* $J(\delta_{p,q}; \nu_p, \nu_q) \in \widehat{G}_{fine}$ *is unitary* $\Rightarrow J_{G_1}(\nu_p) \in \widehat{G_1}_{sph}$ *and* $J_{G_2}(\nu_q) \in \widehat{G_2}_{sph}$.

# Main Theorem

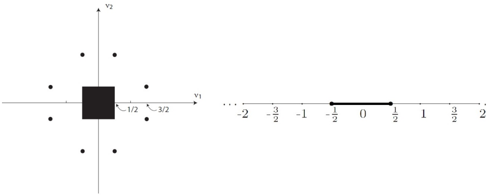
**Conjecture (Barbasch-Ciubotaru, Pantano-Paul-Salamanca)**

*Retain the above setting, then the unitary duals on both sides are equal:*

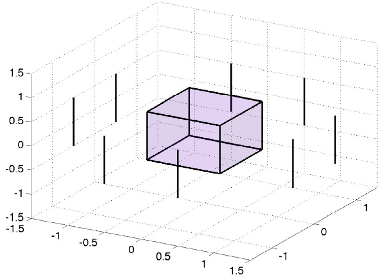
$$J(\delta_{p,q}; \nu_p, \nu_q) \in \widehat{G}_{fine} \text{ is unitary} \Leftrightarrow J_{G_1}(\nu_p) \in \widehat{G_2}_{sph} \text{ and } J_{G_2}(\nu_q) \in \widehat{G_2}_{sph}.$$

**Theorem (Li-W.)**

*The above conjecture holds for* $G = O(n, n+1)$.




Unitary dual of $O(3,2)$ and $O(2,1)$




Unitary dual of $J(\delta_{2,1}; \nu_p, \nu_q)$ in $O(4,3)$

Thank you!