Vertex Lie Bialgebras.

(work in progress)

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Contents.

§1. Recollection : Vertex Algebra.

Def.

A <u>vertex algebra</u> $V = (V, |0\rangle, T, Y)$ consists of the data:

- V : vector space (space of states),
- $|0\rangle \in V$ (vaccum vector),
- $T : V \longrightarrow V$ (translation op.),
- $Y(-, z) = Y(z) : V \otimes V \longrightarrow V((z))$ (state-field correspondence)

satisfying some axioms.

Operator Product Expansion (OPE)

$$[Y(a, z), Y(b, w)] = \sum_{n \geq 0} \left(\frac{\partial_w^n}{n!} \delta(z-w) \right) Y(a_{(n)} b, w).$$

where $\delta(z-w) := \sum_{n=-\infty}^{\infty} z^n w^{-n-1}$.

Exm.

$\mathfrak{g}$: Lie algebra w/ sym. inv. form $(\ ,\ )$.

$\rightsquigarrow$ • $\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t^{\pm}] \oplus \mathbb{C}K$, w/ a Lie bracket given by

$$\begin{cases} [x \otimes t^n, y \otimes t^m] := [x, y] \otimes t^{n+m} + n\,\delta_{n+m,0}\,(x, y)K, \\ K : \text{central}. \end{cases}$$

• $\hat{\mathfrak{g}}_+ := \langle x \otimes t^n \mid x \in \mathfrak{g},\ n \geq 0 \rangle_{\mathbb{C}}$: Lie subalg. of $\hat{\mathfrak{g}}$.

• $V(\mathfrak{g}) := \mathrm{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} \mathbb{C} = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+)} \mathbb{C}$: $\hat{\mathfrak{g}}$-module.

Then, one can obtain VAs :

$V(\mathfrak{g})$: <u>universal affine vertex algebra</u>

$V_k(\mathfrak{g}) = V(\mathfrak{g}) \big/ U(\hat{\mathfrak{g}})(K-k)|0\rangle$: <u>univ. aff. vertex alg. of level $k \in \mathbb{C}$.</u>

This construction can be generalized to vertex Lie algebras.

<u>Def.</u>

A <u>vertex Lie algebra</u> $L := (L, T, Y_-)$ consists of the data:

- L : vector space,
- $T : L \longrightarrow L$
- $Y_-(-, z) = Y_-(z) : L \otimes L \longrightarrow L[z^{-1}]$,

satisfying some axioms.

L : VLA $\rightsquigarrow$ $V(L)$: <u>univ. enveloping VA</u> ([Kac 98], [Prime 99])

<u>Exm.</u>

- V : VA $\rightsquigarrow$ $(V, T, Y_-(z) :=$ (polar part of $Y(z)))$: VLA
- $\mathfrak{g}$: Lie alg. w/ sym. inv. form $(\ ,\ )$.

  Then, one can define a VLA str. on $\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathbb{C}K$

  $\rightsquigarrow$ Its univ. enveloping VA is univ. aff. VA $V(\mathfrak{g})$.

Classical

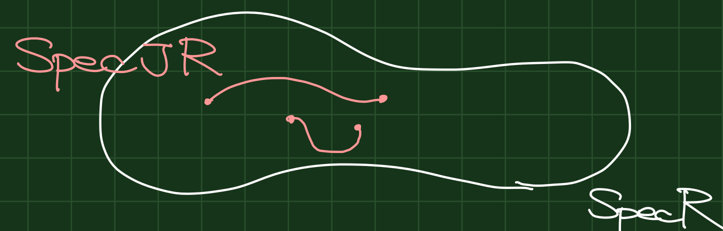
$\mathfrak{g}$: Lie alg.

$U(\mathfrak{g}) \overset{\text{PBW filt.}}{\rightsquigarrow} \mathrm{Sym}\,\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]$: Poisson alg. (Killirov – Kostant Poisson str.)

VA analogue ([Li 04], [Li 05])

V : VA $\overset{\text{Li filt.}}{\rightsquigarrow} \mathrm{gr}\,V$: vertex Poisson algebra (comm. alg. w/ derivation + VLA)

What is a VPA ?



Fact ([Ara 09])

R : PA,

JR : comm. alg. s.t. $\forall S$, $\mathrm{Hom}(JR, S) \simeq \mathrm{Hom}(R, S[\![t]\!])$

$\rightsquigarrow JR$: VPA s.t. $a_{(0)}b = \{a, b\}$ for $a, b \in R \subset JR$

§2. Jet Algebras of Poisson Lie Groups.

aff. alg. grp. (in this talk)

G : aff. alg. grp. $\rightsquigarrow$ $A = \mathbb{C}[G]$: comm. Hopf alg.

Moreover,

G : Poisson aff. alg. grp. $\rightsquigarrow$ A : Poisson Hopf alg.

<u>Exm.</u>

G : semisimple aff. alg. grp..

$\exists$ $\mathfrak{g}$: coboundary Lie bialg. ([CP94, Exm. 2.1.7])

$\rightsquigarrow$ G has the induced Poisson Lie str.. ([CP94, Prop. 2.2.1])

A : Poisson Hopf alg.

$\rightsquigarrow$
- JA : VPA,
- $\Delta : JA \longrightarrow JA \otimes JA$: hom. of VPAs,
- $\varepsilon : JA \longrightarrow \mathbb{C}$: hom. of VPAs ($\mathbb{C}$: triv. VPA)
- $S : JA \longrightarrow JA$: linear map.

such that $(JA, \Delta, \varepsilon)$ is a coalg., and

$$
\begin{tikzcd}
 & JA \otimes JA \arrow[rr, "S \otimes \mathrm{id}"] & & JA \otimes JA \arrow[dr] & \\
JA \arrow[ur] \arrow[rr, "\varepsilon"'] \arrow[dr] & & \mathbb{C} \arrow[rr] & & JA \\
 & JA \otimes JA \arrow[rr, "\mathrm{id} \otimes S"'] & & JA \otimes JA \arrow[ur] &
\end{tikzcd}
$$

Def.

A vertex Poisson Hopf algebra $V = (V, \Delta, \varepsilon, S)$ consists of the data:

- (i) V : VPA,
- (ii) $\Delta : V \longrightarrow V \otimes V$: hom. of VPAs,
- (iii) $\varepsilon : V \longrightarrow \mathbb{C}$: hom. of VPAs,
- (iv) $S : V \longrightarrow V$: linear map,

satisfying the axiom similar to Hopf algebra.

Rmk.

Although S is merely a linear map, the following holds:

$$S(ab) = S(a)S(b), \quad S(Y_-(a,z)b) = -Y_-(S(a),z)S(b).$$

§3. Vertex Lie bialgebra.

V : VPHA.

Since V has a comm. Hopf alg. str., we can consider its Lie algebra :

$$L(V) := \mathrm{Ker}\left(\mathrm{Hom}_{\mathbb{C}\text{-alg}}(V, \mathbb{C}[\epsilon]) \xrightarrow{\epsilon \mapsto 0} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(V, \mathbb{C})\right)$$

$$[x, y](a) := x(a^1)\, y(a^2) - x(a^2)\, y(a^1),$$

where $\Delta(a) = a^1 \otimes a^2$ (Sweedler notation).

Exm.

$A = \mathbb{C}[G]$, $V = JA$.

Then,

$$\mathfrak{g}[\![t]\!] \simeq L(V) = \mathrm{Ker}\left(\mathrm{Hom}_{\mathbb{C}\text{-alg}}(A, \mathbb{C}[\![t]\!][\epsilon]) \xrightarrow{\epsilon \mapsto 0} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(A, \mathbb{C}[\![t]\!])\right)$$

Furthermore, $L(V)$ should carry a str. induced by $Y_-(z)$.

<u>Technical assumption</u>

- V : $\mathbb{Z}_{\geq 0}$-gr. VPA, i.e. $V = \bigoplus_{h \geq 0} V_h$ w/ some condition
- $L = L^{fin}(V) := L(V) \cap \{ V \xrightarrow{\phi} \mathbb{C}[\varepsilon] \mid \phi(V_{\gg 0}) = 0 \}$,

Then, L has a natural $\mathbb{Z}_{\geq 0}$-grading : $L = \bigoplus_{h \geq 0} L_h$.

- $\dim L_h < \infty$.

<u>Exm.</u>

$V = J\mathbb{C}[G] \rightsquigarrow L^{fin}(V) = \mathfrak{g}[t] = \bigoplus_{h \geq 0} \mathbb{C}\, \mathfrak{g} t^h$

$$V \otimes V \xrightarrow{Y_-(z)} V[z^{-1}]$$

$$\rightsquigarrow \quad \begin{array}{ccc} L & \xrightarrow{\delta(z)} & L \otimes L [\![z^{-1}]\!] \\ x & \longmapsto & (a \otimes b \longmapsto x(Y_-(a,z)b)) \end{array}$$

Then, we can verify

(i) The restricted dual $L^\vee = \bigoplus_{n \geq 0} L_n^*$ has a VPA str.,

(ii) The compatible condition (cocycle condition):

$$\delta([x,y],z) = \mathrm{Sing}\begin{pmatrix} [e^{zT}x \otimes 1 + 1 \otimes x,\ \delta(y,z)] \\ -[e^{zT}y \otimes 1 + 1 \otimes y,\ \delta(x,z)] \end{pmatrix}$$

<u>Exm.</u>

$\mathfrak{g}$: Lie bialg., $L = \mathfrak{g}[t]$.

Then,

$$\delta(xt^n)(z) := \frac{t_2^n}{z-(t_2-t_1)}\delta(x) = \sum_{n \geq 0} \frac{t_2^n (t_2-t_1)^n}{z^{n+1}}\delta(x),$$

where $t_1 = t \otimes 1$, $t_2 = 1 \otimes t$ in $\mathfrak{g}[t] \otimes \mathfrak{g}[t]$.

Def.

A <u>vertex Lie bialgebra</u> $L = (L, T, \delta)$ consists of the data :

- (L, T) : Lie algebra w/ derivation,
- $\delta(z) = \delta(-, z) : L \longrightarrow L \otimes L[\![z^{-1}]\!]$,

such that

(i) (L, T, δ) : vertex co-Lie algebra,

(ii) compatible condition (cocycle condition) :

$$\delta([x, y], z) = \mathrm{Sing}\begin{pmatrix} [e^{zT}x \otimes 1 + 1 \otimes x, \ \delta(y, z)] \\ - [e^{zT}y \otimes 1 + 1 \otimes y, \ \delta(x, z)] \end{pmatrix}$$

§4. Quantimization

The story of quantum groups

$U_q(\mathfrak{g})$: quasitriangular Hopf alg.

$$\Delta^{op}(x) = R\Delta(x)R^{-1},$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

$\Big\downarrow$ $q \to 1$ (or $\hbar \to 0$)

$\mathfrak{g}$: coboundary Lie bialg.

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

?

Candidates

- VCA [Hub]
- QVA [EK]

$\Big\downarrow$

L : coboundary VLBA

Thank you for your attention.