On the Pieri formula of super Macdonald polynomials

Based on a collaboration with R.Ohkawa and J.Shiraishi,
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Hiroaki Kanno

Graduate School of Mathematics & Kobayashi-Maskawa Institute, Nagoya University

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Macdonald polynomials

$\Lambda_n := \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$: Ring of the symmetric polynomials in $x_1, \ldots, x_n$.

$$\Lambda_n \ni f(x_1, x_2, \ldots, x_n), \qquad f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}), \quad \sigma \in S_n.$$

We may extend the coefficients to any field $\mathbb{F}$ to define $\Lambda_{\mathbb{F}} := \Lambda_n \otimes_{\mathbb{Z}} \mathbb{F}$, for example $\mathbb{F} = \mathbb{Q}$.

The Macdonald polynomials $P_\lambda(x; q, t)$ give a distinguished basis of $\Lambda_{\mathbb{F}}$ with $\mathbb{F} = \mathbb{Q}(q, t)$; the field of rational functions in parameters (q, t), indexed by the partitions λ.

$$P_{\square\square}(x; q, t) = \frac{1}{2}\left(\frac{(1+q)(1-t)}{1-qt} p_1^2 + \frac{(1+t)(1-q)}{1-qt} p_2\right), \qquad P_{\substack{\square\\\square}}(x; q, t) = \frac{1}{2}(p_1^2 - p_2),$$

where $p_k(x) := \sum_{i=1}^n x_i^k$ is the power sum symmetric polynomial.

1. When $q = t$, $P_\lambda(x; q, t)$ reduce to the Schur polynomials $s_\lambda(x)$.
2. The Macdonald polynomials $P_\lambda(x; q, t)$ are invariant under $(q, t) \to (q^{-1}, t^{-1})$.

Pieri formula

The Pieri formula for $P_{\Box} = p_1 = x_1 + x_2 + \cdots$;

$$p_1 \cdot P_\lambda(x;q,t) = \sum_{\mu} \psi'_{\mu/\lambda}(q,t) P_\mu(x;q,t),$$

where the sum is over the partitions μ such that $|\mu| - |\lambda| = 1,\ \mu_i - \lambda_i \leq 1$.
The expansion coefficients are

$$\psi'_{\mu/\lambda}(q,t) = \prod_{i=1}^{j-1} \frac{(1-q^{\lambda_i-\lambda_j}t^{j-i-1})(1-q^{\lambda_i-\lambda_j-1}t^{j-i+1})}{(1-q^{\lambda_i-\lambda_j}t^{j-i})(1-q^{\lambda_i-\lambda_j-1}t^{j-i})}, \qquad \mu_k = \lambda_k + \delta_{kj}.$$

$$P_{\Box}(x;q,t) \cdot P_{\Box}(x;q,t) = p_1^2 = P_{\Box\Box}(x;q,t) + \frac{(1-q)(1-t^2)}{(1-qt)(1-t)} P_{\substack{\Box\\\Box}}(x;q,t).$$

When $q = t$, namely for Schur polynomials $s_\lambda(x)$, $\psi'_{\mu/\lambda}(q,q) = 1$.

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Super Macdonald polynomials

The super Macdonald polynomials $\mathcal{M}_\Lambda(x,\theta;q,t)$ generalize the Macdonald polynomials to the super space with additional Grassmann coordinates θ_i. They are invariant under *simultaneous* exchanges of the commuting and the anti-commuting coordinates $(x_i,\theta_i) \to (x_{\sigma(i)},\theta_{\sigma(i)})$.

The super Macdonald polynomials $\mathcal{M}_\Lambda(x,\theta;q,t)$ are indexed by the set of super partitions; A super partition is a non-increasing sequence of non-negative elements in $\mathbb{Z}/2$;

$$\Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_i \geq \ldots \geq \Lambda_{\ell(\Lambda)} > 0, \qquad \Lambda_i \in \mathbb{Z}_{\geq 0}/2,$$

where $\ell(\Lambda)$ is the number of non-zero components in Λ. $|\Lambda| = \sum_{i=1}^{\ell(\Lambda)} \Lambda_i$ is called level of Λ.

$\mathbb{Z}_2$ grading of $\mathbb{Z}/2$: integral elements are even and non-integral elements are odd.
We require that if Λ_k is odd ($\Lambda_k \in \mathbb{Z} + \frac{1}{2}$), the inequality is strict $\Lambda_{k-1} > \Lambda_k > \Lambda_{k+1}$.

Super partition and super Young diagram

Identify a super partition with a super Young diagram, which consists of full boxes and marked boxes by $\times$. If $\Lambda_i \in \mathbb{Z}$, the end of the corresponding row is a full box and if $\Lambda_i \in \mathbb{Z} + \frac{1}{2}$, the end of the corresponding row is a marked box which is counted as $\frac{1}{2}$.

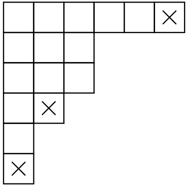

Figure: Super partition $\Lambda = (\frac{11}{2}, 3, 3, \frac{3}{2}, 1, \frac{1}{2})$ with fermion number 3.

For each row and each column the number of the marked box is at most one.

Another description of super partitions is $\Lambda = (\lambda, \overline{\sigma})$ by a partition λ and $\overline{\sigma} = (\sigma_i)$ with $\sigma_i \in \{0, 1\}$. We define $\Lambda_i = \lambda_i - \frac{1}{2}\sigma_i$. The row is bosonic or fermionic according to $\sigma_k = 0, 1$.

Super Macdonald polynomials up to $|\Lambda| \leq 2$

In terms of the bosonic and the fermionic power sum polynomials defined by

$$p_k := \sum_{i=1}^{n} x_i^k, \qquad \pi_k := \sum_{i=1}^{n} \theta_i x_i^{k-1},$$

we have

$$\mathcal{M}_{(\frac{1}{2})} = \pi_1, \qquad \mathcal{M}_{(1)} = p_1, \qquad \mathcal{M}_{(\frac{3}{2})} = \frac{q(1-t)}{1-qt} p_1 \pi_1 + \frac{1-q}{1-qt} \pi_2,$$

$$\mathcal{M}_{(1,\frac{1}{2})} = p_1 \pi_1 - \pi_2, \qquad \mathcal{M}_{(2)} = \frac{1}{2} \left(\frac{(1+q)(1-t)}{1-qt} p_1^2 + \frac{(1+t)(1-q)}{1-qt} p_2 \right),$$

$$\mathcal{M}_{(1,1)} = \frac{1}{2}(p_1^2 - p_2), \qquad \mathcal{M}_{(\frac{3}{2},\frac{1}{2})} = \pi_2 \pi_1.$$

1. Even if we set $q = t$, there remains one parameter in the super "Schur" polynomials.
2. The super Macdonald polynomials $\mathcal{M}_\Lambda(x, \theta; q, t)$ are not invariant under $(q,t) \to (q^{-1}, t^{-1})$.

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Quantum toroidal algebra of type $\mathfrak{gl}_{1|1}$

One of the ways to define the super Macdonald polynomials is to use the representation theory of the quantum toroidal algebra $\mathcal{U}_{(q,t)}(\widehat{\widehat{\mathfrak{gl}}}_{1|1})$, which is a quantum group (a quasi-triangular Hopf superalgebra) with generators $E_{i,k}, F_{i,k}, K^{\pm}_{i,\pm r}$ $(i = 1, 2)$ and a central element C. $E_{i,k}$ and $F_{i,k}$ are odd generators. The algebra has two deformation parameters (q, t).

It is convenient to introduce the generating currents;

$$E_i(z) = \sum_{k\in\mathbb{Z}} E_{i,k} z^{-k}, \qquad F_i(z) = \sum_{k\in\mathbb{Z}} F_{i,k} z^{-k},$$

$$K^{\pm}_i(z) = \sum_{r\geq 0} K^{\pm}_{i,\pm r} z^{\mp r} = K^{\pm}_{i,0} \cdot \exp\left(\pm \sum_{r=1}^{\infty} H_{i,\pm r} z^{\mp r} \right).$$

Level zero representation

For an irreducible representation with $C = 1$, we say it has level zero. In this case, $K_i^{\pm}(z)$ are mutually commuting. Hence, in this case we can choose a basis which simultaneously diagonalizes $K_{i,\pm r}^{\pm}$ or $H_{i,\pm r}$. Some of the commutation relations are

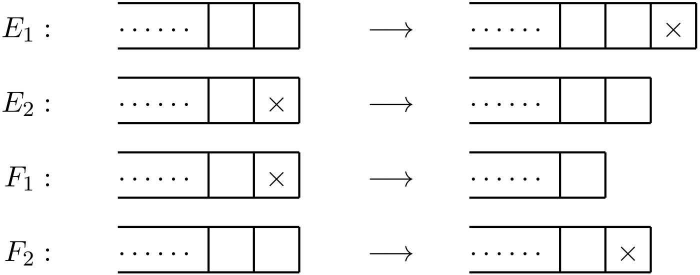
$$[H_{i,r}, E_{j,s}] = +\epsilon_{ij}\frac{1}{r}(q^{\frac{r}{2}} - q^{-\frac{r}{2}})(t^{\frac{r}{2}} - t^{-\frac{r}{2}})E_{j,r+s}, \qquad \epsilon_{12} = -\epsilon_{21} = 1,$$
$$[H_{i,r}, F_{j,s}] = -\epsilon_{ij}\frac{1}{r}(q^{\frac{r}{2}} - q^{-\frac{r}{2}})(t^{\frac{r}{2}} - t^{-\frac{r}{2}})F_{j,r+s}.$$

$$[E_i(z), F_{j,s}]_+ = \delta_{ij}\ z^s \left(z^{r_i} K_i^+(z) - K_i^-(z)\right).$$

When the shift parameter r_i is non-vanishing, we say it is a shifted algebra.
The shift parameter is fixed by the asymptotic behavior of $[E_i(z), F_{j,s}]_+$ at $z \to \infty$.

Super Fock representation

We can construct a level zero representation on the vector space spanned by super partitions. The action of odd currents $E_i(z)$ and $F_i(z)$ on the super partitions is described as follows;



$$E_{1,2} : |\Lambda| \longrightarrow |\Lambda| + \tfrac{1}{2} \text{ and } F_{1,2} : |\Lambda| \longrightarrow |\Lambda| - \tfrac{1}{2}.$$

E_1, F_2 : boson $\longrightarrow$ fermion, while E_2, F_1 : fermion $\longrightarrow$ boson.

Generating functions of the eigenvalues

In the super Fock representation the Cartan generators $K^{\pm}_{i,\pm r}$ are simultaneously diagonalized on the set of super partitions. The Cartan currents $K^{\pm}_i(z)$ are nothing but the generating functions of the eigenvalues.

$$K^{\pm}_1(z) = \left[\frac{1}{z - t^{-\ell(\Lambda)}} \prod_{i=1}^{\ell(\Lambda)} \frac{z - q^{\lambda_i - \overline{\sigma}_i} t^{-i}}{z - q^{\lambda_i - \overline{\sigma}_i} t^{1-i}} \right]_{\pm},$$

$$K^{\pm}_2(z) = \left[(z - (t/q)^{\frac{1}{2}} t^{-\ell(\Lambda)}) \prod_{i=1}^{\ell(\Lambda)} \frac{z - q^{\lambda_i - \frac{1}{2}} t^{\frac{3}{2} - i}}{z - q^{\lambda_i - \frac{1}{2}} t^{\frac{1}{2} - i}} \right]_{\pm},$$

where for a rational function $f(z)$, $[f(z)]_{\pm}$ denotes the (Laurent) expansion around $z = \infty$ or $z = 0$. We see the shift parameters are $r_1 = -1$ and $r_2 = 1$.

Main Conjecture: [Pieri rule of the super Macdonald polynomials]

Under the linear isomorphism $|\Lambda\rangle \mapsto \mathcal{M}_\Lambda(x,\theta;q,t)$, the action of the quantum toroidal algebra is realized as linear differential operators on p_k and π_k.
In particular, the following four modes take a simple form;

$$E_{1,0} = \pi_1, \quad E_{2,0} = \sum_{k=1}^{\infty} c_k[p] \frac{\partial}{\partial \pi_k}, \qquad \sum_{k=0}^{\infty} c_k[p] z^k = \exp\left(\sum_{r=1}^{\infty} \frac{1-t^r}{r} p_r z^r \right).$$

$$F_{1,+1} = \frac{\partial}{\partial \pi_1}, \qquad F_{2,-1} = -\sum_{k=1}^{\infty} \pi_k \cdot \widetilde{c}_k \left[\partial/\partial p\right],$$

$$\sum_{k=0}^{\infty} \widetilde{c}_k \left[\partial/\partial p\right] z^{-k} = \exp\left(\sum_{n=1}^{\infty} (q^{-n}-1) \frac{\partial}{\partial p_n} z^{-n} \right).$$

We take not $F_{1,0}$ and $F_{2,0}$, but $F_{1,+1}$ and $F_{2,-1}$, since the toroidal algebra is shifted.
We have checked the conjecture up to level four $|\Lambda| \leq 4$.

Consequence of the conjecture

The anti-commutation relation

$$[E_i(z), F_j(w)]_+ = \delta_{ij}\delta\left(\frac{w}{z}\right)\left(z^{r_i}K_i^+(z) - K_i^-(z)\right),$$

implies

$$[E_{2,0}, F_{2,-1}]_+ = 1 - (t/q)^{\frac{1}{2}} H_{2,-1}, \qquad [E_{1,-1}, F_{1,0}]_+ = -H_{1,-1}.$$

Assuming the conjecture, we can compute the first Hamiltonians $H_{2,-1}$ and $H_{1,-1}$ of the mutually commuting Hamiltonians $H_{i,-r}$ $(r > 0)$. For example,

$$\begin{aligned}(t/q)^{\frac{1}{2}} H_{2,-1} &= 1 + \left[\sum_{k=1}^{\infty} c_k[p]\frac{\partial}{\partial \pi_k}, \sum_{\ell=1}^{\infty} \pi_\ell \cdot \widetilde{c}_\ell\,[\partial/\partial p]\right]_+ \\ &= 1 + \sum_{k=1}^{\infty} c_k[p]\widetilde{c}_\ell\,[\partial/\partial p] + \sum_{k,\ell=1}^{\infty} C_{k,\ell}[p, \partial/\partial p]\pi_\ell \frac{\partial}{\partial \pi_k}.\end{aligned}$$

$$C_{k,\ell}[p, \partial/\partial p] := \left[\widetilde{c}_\ell[\partial/\partial p], c_k[p]\right].$$

The bosonic part

$$(t/q)^{\frac{1}{2}} H_{2,-1}|_{\pi_k=0} = 1 + \sum_{k=1}^{\infty} c_k[p]\widetilde{c}_k\,[\partial/\partial p]$$
$$= \oint \frac{dz}{2\pi i z} \exp\left(\sum_{r=1}^{\infty} \frac{1-t^r}{r} p_r z^r\right) \exp\left(\sum_{n=1}^{\infty} (q^{-n}-1)\frac{\partial}{\partial p_n} z^{-n}\right)$$

agrees with the result of Awata, Kubo, Odake and Shiraishi (1996).

On the other hand, the fermion bi-linear part agrees with a recent proposal by Galakhov, Morozov and Tselousov (2025).

Similarly we obtain $H_{1,-1} = 1 + \sum_{k=1}^{\infty} c_k[p]\widetilde{c}_k\,[\partial/\partial p] + \sum_{k,\ell=2}^{\infty} C_{k-1,\ell-1}[p,\partial/\partial p]\pi_\ell \frac{\partial}{\partial \pi_k}$.
The difference from $H_{2,-1}$ is only the fermionic bilinear terms.

The Hamiltonians $H_{2,+1}$ and $H_{1,+1}$ are much more involved, There appear higher order terms in fermions. This should be related to the fact that $\mathcal{M}_\Lambda(x,\theta;q^{-1},t^{-1}) \neq \mathcal{M}_\Lambda(x,\theta;q,t)$.