The State-Refocusing Square-Root Instrument and Retrodictive Entropy Uncertainty Relations

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Overview

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Background

- Quantum measurement shows one of the main differences between classical and quantum physics.
- General quantum measurement is always represented as Positive Operator-Valued Measure (POVM) $\mathbf{M} = \{M_x\}_{x \in \mathcal{X}}$, and in quantum information theory, the measurement process can also be expressed by a quantum instrument.
- In the study of quantum measurement, one of the most important topics is the study of uncertainty relations, which has been studied extensively these days.
- Start from a special kind of quantum instrument that we call "State-Refocusing Square Root Instrument" (SRSR-Instrument), we focus on its properties and its possible applications on uncertainty relations.

Positive Operator-Valued Measurement

Definition of Positive Operator-Valued Measurement (POVM) and the measuring output possibility distribution are shown below:

Positive Operator-Valued Measurement (POVM)

A positive operator-valued measure (POVM) is a set of several positive operators $\mathbf{M} = \{M_x\}_{\mathcal{X}}$ with $x \in \mathcal{X}$ the outcome set such that:

$$M_x \geq 0, \ \forall x \in \mathcal{X} \tag{1}$$

$$\sum_x M_x = \mathbb{1}. \tag{2}$$

Let the quantum state be denoted by a density matrix ρ, the probability of obtaining the outcome x follows the Born rule:

$$\Pr(x) = \mathrm{Tr}[M_x \rho] \tag{3}$$

Quantum Instrument

When describing the whole quantum measurement process, it is convenient to introduce the quantum instrument.

Quantum Instrument

Let $\mathcal{H}_{\text{in}}, \mathcal{H}_{\text{out}}$ be finite-dimensional Hilbert spaces. A quantum instrument is a collection $\{\mathcal{J}_x^M\}_{\mathcal{X}}$ with each $\mathcal{J}_x^M : \mathcal{B}(\mathcal{H}_{\text{in}}) \to \mathcal{B}(\mathcal{H}_{\text{out}})$ a completely positive trace non-increasing (CPTNI) map and the sum of them is trace preserving (TP) corresponding to a given POVM $\mathbf{M} = \{M_x\}_{\mathcal{X}}$ such that:

$$\text{Tr}[\mathcal{J}_x^{\mathbf{M}}(\rho)] = \text{Tr}[M_x \rho]. \tag{4}$$

The above implies that $\mathcal{J}^{\mathbf{M}}(\cdot) : \rho \to \sum_x \mathcal{J}_x^{\mathbf{M}}(\rho) \otimes |x\rangle \langle x|$, where $\{|x\rangle\}$ is an orthonormal basis labelling the classical output of the instrument, is a CPTP map.

Quantum Instrument

- **Kraus Representation:**
  A linear completely positive and trace-preserving (CPTP) map $\mathcal{E} : \mathcal{B}(\mathcal{H}_{\text{in}}) \to \mathcal{B}(\mathcal{H}_{\text{out}})$ admits a Kraus representation $\mathcal{E}(\rho) = \sum_{i=1}^{r} K_i \, \rho \, K_i^\dagger$, where $K_i : \mathcal{H}_{\text{in}} \to \mathcal{H}_{\text{out}}$ are so-called Kraus operator with the trace-preserving condition $\sum_{i=1}^{r} K_i^\dagger K_i = \mathbb{1}_{\text{in}}$.
- **Square Root Instrument:**
  A square-root instrument $\{\mathcal{J}_x^{\mathbf{M}}\}_{\mathcal{X}}$, also known as the efficient instrument, is defined as follows:

$$\mathcal{J}^{\mathbf{M}}(\cdot) : \rho \to \sum_x \sqrt{M_x} \rho \sqrt{M_x} \otimes |x\rangle \langle x| \,, \tag{5}$$

where $\sqrt{M_x}^\dagger \sqrt{M_x} = M_x$. It is efficient because there is only one Kraus operator $\sqrt{M_x}$ for each outcome x. One can easily check that this instrument coincide with 4.

Notice that in general, one can always do the post-processing with a unitary U_x for each outcome x after the quantum instrument (or, one can think that it is the general square-root instrument) such that:

$$\mathcal{I}^{\mathbf{M}}(\cdot) : \rho \rightarrow \sum_x U_x \sqrt{M_x} \rho \sqrt{M_x} U_x^\dagger \otimes |x\rangle \langle x| . \tag{6}$$

Consider the polar decomposition $\sqrt{\rho}\sqrt{M_x} = U_x^\rho \sqrt{\sqrt{M_x}\sqrt{\rho}\sqrt{\rho}\sqrt{M_x}} = U_x^\rho \sqrt{\sqrt{M_x}\rho\sqrt{M_x}}$, one can see that:

$$\sqrt{\rho} M_x \sqrt{\rho} = \sqrt{\rho}\sqrt{M_x}\sqrt{M_x}\sqrt{\rho} = U_x^\rho \sqrt{M_x} \rho \sqrt{M_x} U_x^{\rho\dagger} . \tag{7}$$

Therefore, by choosing the proper U_x^ρ which is based on the state ρ and the measurement M_x, we have state-refocusing square-root instrument:

State-Refocusing Square Root Instrument

$$\mathcal{I}_\rho^{\mathbf{M}}(\rho) = \sum_x U_x^\rho \sqrt{M_x} \rho \sqrt{M_x} U_x^{\rho\dagger} \otimes |x\rangle \langle x| = \sum_x \sqrt{\rho} M_x \sqrt{\rho} \otimes |x\rangle \langle x| . \tag{8}$$

Properties of SRSR-Instrument

- Petz transpose map: The well known *Petz transpose map* of a map $\mathcal{E}$ with the prior state γ is:

$$\mathcal{R}(\omega) = \sqrt{\gamma}\mathcal{E}^{\dagger}(\mathcal{E}(\gamma)^{-\frac{1}{2}}\omega\mathcal{E}(\gamma)^{-\frac{1}{2}})\sqrt{\gamma}. \tag{9}$$

Now, for the SRSR-Instrument defined in 8, define the correspondent quantum-classical (QC) channel $\mathcal{M} : \rho \rightarrow \sum_x \mathrm{Tr}[M_x\rho] |x\rangle \langle x|$. One can easily see $\mathcal{M}(\cdot) = \mathrm{Tr}_Q[\mathcal{I}_\rho^{\mathbf{M}}(\cdot)]$, where the subscript Q in Tr_Q refers to the "quantum part" of the output state of $\mathcal{I}_\rho^{\mathbf{M}}$. For each outcome $x_0 \in \mathcal{X}$, let $\mathcal{E} = \mathcal{M}$, $\omega = |x_0\rangle \langle x_0|$, and set the prior state $\gamma = \rho$. We can get the output state of Petz transpose map $\mathcal{R}_\rho^{\mathbf{M}}$ is:

Petz transpose map and SRSR-Instrument (On the corresponding QC-Channel)

$$\begin{aligned}\mathcal{R}_\rho^{\mathbf{M}}(|x_0\rangle \langle x_0|) &= \sqrt{\rho}\mathcal{M}^{\dagger}(\mathcal{M}(\rho)^{-\frac{1}{2}} |x_0\rangle \langle x_0| \mathcal{M}(\rho)^{-\frac{1}{2}})\sqrt{\rho} \\ &= \sqrt{\rho}\mathcal{M}^{\dagger}(\frac{|x_0\rangle \langle x_0|}{\mathrm{Tr}[M_{x_0}\rho]})\sqrt{\rho} \\ &= \frac{\sqrt{\rho}M_{x_0}\sqrt{\rho}}{\mathrm{Tr}[M_{x_0}\rho]}.\end{aligned} \tag{10}$$

Properties of SRSR-Instrument

Petz transpose map and SRSR-Instrument (On the corresponding QC-Channel)

$$\mathcal{R}_\rho^{\mathbf{M}}(|x_0\rangle\langle x_0|) = \frac{\sqrt{\rho}M_{x_0}\sqrt{\rho}}{\mathrm{Tr}[M_{x_0}\rho]}$$

To calculate the $\mathcal{M}^\dagger$ one can consider the Kraus operator of $\mathcal{M}$ is $K_{i,x} = |x\rangle\langle i|\sqrt{M_x}$.

- We found that the output of the Petz transpose map $\mathcal{R}_\rho^{\mathbf{M}}$ for one output information (the classical outcome x_0, e.g.) is the normalized state of the "quantum part" of the output state of our SRSR-Instrument for that classical outcome.
- The question: What does the Petz transpose map mean?

Quantum Bayes' Rule and Retrodiction

Let $Q_{\text{fwd}}, Q_{\text{rev}}$ be the bipartite states representing forward channel $\mathcal{E}$ and the reverse channel $\mathcal{R}$ respectively:

$$
\begin{aligned}
Q_{\text{fwd}} &:= \mathcal{E} \star \rho = (\mathcal{E} \otimes id)(|\sqrt{\rho}\rangle\rangle\langle\langle\sqrt{\rho}|) \\
&= (\mathbb{1}_{out} \otimes \sqrt{\rho^{\mathsf{T}}})C_{\mathcal{E}}(\mathbb{1}_{out} \otimes \sqrt{\rho^{\mathsf{T}}}) \qquad \in \mathcal{S}(\mathcal{H}_{out} \otimes \mathcal{H}_{in}), \qquad (11) \\
Q_{\text{rev}} &:= (\mathcal{R} \star \tau)^{\mathsf{T}} = (id \otimes \mathcal{R})^{\mathsf{T}}(|\tau\rangle\rangle\langle\langle\tau|) \\
&= (\sqrt{\tau} \otimes \mathbb{1}_{in})C_{\mathcal{R}}^{\mathsf{T}}(\sqrt{\tau} \otimes \mathbb{1}_{in}) \qquad \in \mathcal{S}(\mathcal{H}_{out} \otimes \mathcal{H}_{in}), \qquad (12)
\end{aligned}
$$

where $|\rho\rangle\rangle := \sum_{ij} \langle i| \sqrt{\rho}\, |j\rangle\, |i\rangle_{A_1} |j\rangle_{A_2}$ is the canonical purification of ρ, and $C_{\mathcal{E}} := \sum_{i,j} \mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j| \in \mathcal{L}_+(\mathcal{H}_{out} \otimes \mathcal{H}_{in})$, $C_{\mathcal{R}} := \sum_{i,j} |i\rangle\langle j| \otimes \mathcal{R}(|i\rangle\langle j|) \in \mathcal{L}_+(\mathcal{H}_{out} \otimes \mathcal{H}_{in})$ are the Choi operators of $\mathcal{E}$ and $\mathcal{R}$.

With the preparation of the definition of the forward and reverse states, we can now construct the quantum minimal change principle.

Quantum Bayes' Rule and Retrodiction

Classical Bayes' rule can be derived from classical minimal change principle. Recently, the quantum minimal change principle is defined by [1] and be used to derive the Quantum Bayes' rule.

Quantum Minimal Change Principle [1]

The quantum minimal change principle is an optimization problem such that

$$\begin{aligned} \max_{\mathcal{R}} \quad & F(\mathcal{E} \star \rho, (\mathcal{R} \star \tau)^{\mathsf{T}}) \\ \text{subject to} \quad & \mathcal{R} \text{ is CPTP.} \end{aligned} \tag{13}$$

Where the $F(\rho, \sigma) := \mathrm{Tr}[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}] = \mathrm{Tr}[\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}]$ is fidelity, which is used to measure the state change.

Quantum Bayes' Rule [1]

For a CPTP map $\mathcal{E}$, suppose the reference prior is γ and the reference output is τ, assuming $\mathcal{E} \star \gamma > 0$ and $\tau > 0$, the following CPTP map

$$\mathcal{R}(\omega) := \sqrt{\gamma}\mathcal{E}^{\dagger}(D\omega D^{\dagger})\sqrt{\gamma} \tag{14}$$

$$D := \sqrt{\tau}(\sqrt{\tau}\mathcal{E}(\gamma)\sqrt{\tau})^{-\frac{1}{2}} \tag{15}$$

is the unique solution of the program 13. Furthermore, if $[\tau, \mathcal{E}(\gamma)] = 0$, then the solution coincide with the *Petz transpose map* 9.

- Notice that when the forward channel $\mathcal{E}$ is a POVM qc-channel, then the reference output τ is the classical information that we got from the POVM, and $[\tau, \mathcal{E}(\gamma)] = 0$, so for the quantum instrument, the quantum Bayes' rule coincide with the Petz transpose map. Recall 10, now we can say that it is exactly a state update rule based on the classical information $x \in \mathcal{X}$.
- As discussed before, SRSR-Instrument output the sub-normalized retrodictive update states after the POVM for each classical outcome x.

Quantum Bayes' Rule and Retrodiction

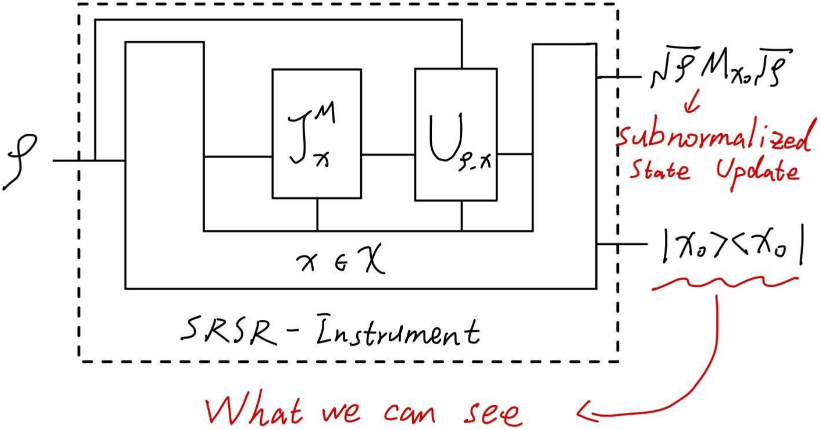

Figure: Description of SRSR-Instrument

- Sequential Measurement:
Base on SRSR-Instrument on the first measurement, one can consider the following sequential measurement of two POVM $\mathbf{M} = \{M_x\}_{\mathcal{X}}$ and $\mathbf{N} = \{N_y\}_{\mathcal{Y}}$:

$$\mathcal{I}^{\mathbf{N}} \circ \mathcal{I}^{\mathbf{M}}_{\rho}(\rho) = \sum_{x,y} \sqrt{N_y}\sqrt{\rho}M_x\sqrt{\rho}\sqrt{N_y} \otimes |x\rangle\langle x| \otimes |y\rangle\langle y| . \tag{16}$$

An interesting observation is that, consider using SRSR-Instrument for the first measurement, then joint probability does not change even swap the order of measurement:

$$\Pr(x,y) = \mathrm{Tr}[\mathcal{I}^{\mathbf{N}}_y \circ \mathcal{I}^{\mathbf{M}}_{x,\rho}(\rho)] = \mathrm{Tr}[N_y\sqrt{\rho}M_x\sqrt{\rho}] = \mathrm{Tr}[M_x\sqrt{\rho}N_y\sqrt{\rho}] = \mathrm{Tr}[\mathcal{I}^{\mathbf{M}}_x \circ \mathcal{I}^{\mathbf{N}}_{y,\rho}(\rho)]. \tag{17}$$

The marginals also correct (One can check):

$$\sum_x \mathrm{Tr}[\mathcal{I}^{\mathbf{N}}_y \circ \mathcal{I}^{\mathbf{M}}_{x,\rho}(\rho)] = \sum_x \mathrm{Tr}[\mathcal{I}^{\mathbf{M}}_x \circ \mathcal{I}^{\mathbf{N}}_{y,\rho}(\rho)] = \mathrm{Tr}[\mathcal{I}^{\mathbf{N}}_y(\rho)], \tag{18}$$

$$\sum_y \mathrm{Tr}[\mathcal{I}^{\mathbf{N}}_y \circ \mathcal{I}^{\mathbf{M}}_{x,\rho}(\rho)] = \sum_y \mathrm{Tr}[\mathcal{I}^{\mathbf{M}}_x \circ \mathcal{I}^{\mathbf{N}}_{y,\rho}(\rho)] = \mathrm{Tr}[\mathcal{I}^{\mathbf{M}}_x(\rho)]. \tag{19}$$

The Connection with the Uncertainty Relations

For the sequential measurement channel, the output state

$$\sigma = \mathcal{I}^{\mathbf{N}} \circ \mathcal{I}^{\mathbf{M}}_{\rho}(\rho) = \sum_{x,y} \sqrt{N_y}\sqrt{\rho}M_x\sqrt{\rho}\sqrt{N_y} \otimes |x\rangle\langle x| \otimes |y\rangle\langle y| , \tag{20}$$

Notice that the joint probability $\Pr(x,y) = \mathrm{Tr}[N_y\sqrt{\rho}M_x\sqrt{\rho}] = \mathrm{Tr}[\sqrt{N_y}\sqrt{\rho}\sqrt{M_x}\sqrt{M_x}\sqrt{\rho}\sqrt{N_y}] = ||\sqrt{N_y}\sqrt{\rho}\sqrt{M_x}||_2^2$, the joint entropy of X and Y have the following inequality:

State-Dependent Entropic Uncertainty Relation 1

$$\begin{aligned} H(X)_\rho + H(Y)_\rho &= H(X)_\sigma + H(Y)_\sigma \\ &\geq H(XY)_\sigma \\ &= -\sum_{x,y} \mathrm{Tr}[N_y\sqrt{\rho}M_x\sqrt{\rho}] \log(\mathrm{Tr}[N_y\sqrt{\rho}M_x\sqrt{\rho}]) \\ &\geq -\max_{x,y} \log ||\sqrt{N_y}\sqrt{\rho}\sqrt{M_x}||_2^2. \end{aligned} \tag{21}$$

The Connection with the Uncertainty Relations

Compare to the famous Maassen and Uffink's entropic uncertainty bound [2]

$$H(X) + H(Y) \geq -\max_{x,y} \log ||\sqrt{M_x}\sqrt{N_y}||_\infty^2, \tag{22}$$

our bound maybe useful for entropic uncertainty principle.
The sequential measurement channel 16 is CPTP, sub-unital. We can thus consider the bound [3]

$$H(XY)_\sigma - I_{GO}(\rho; \mathcal{I}^{\mathbf{N}} \circ \mathcal{I}^{\mathbf{M}}_\rho) \geq D(\rho||\tilde{\rho}), \tag{23}$$

where

$$\tilde{\rho} := ((\mathcal{I}^{\mathbf{N}} \circ \mathcal{I}^{\mathbf{M}}_\rho)^\dagger \circ (\mathcal{I}^{\mathbf{N}} \circ \mathcal{I}^{\mathbf{M}}_\rho))(\rho),$$

$$I_{GO}(\rho; \mathcal{I}^{\mathbf{N}} \circ \mathcal{I}^{\mathbf{M}}_\rho) = H(\rho) - \sum_{x,y} \Pr(x,y) H(\mathcal{I}^{\mathbf{N}}_y \circ \mathcal{I}^{\mathbf{M}}_{x,\rho}(\rho)).$$

Note that $(\mathcal{I}^{\mathbf{N}} \circ \mathcal{I}^{\mathbf{M}}_\rho)^\dagger$ is trace-non-increasing, so $\tilde{\rho}$ is subnormalized. Since $B \leq B' \Rightarrow D(A||B) \geq D(A||B')$, the use of $\tilde{\rho}$ actually makes the bound tighter.
Assume now the two POVMs are both rank-one POVM, in this case, $I_{GO}(\rho; \mathcal{I}^{\mathbf{N}} \circ \mathcal{I}^{\mathbf{M}}_\rho) = H(\rho)$. We then have (see the next page):

The Connection with the Uncertainty Relations

State-Dependent Entropic Uncertainty Relation 2

$$
\begin{aligned}
H(X)_\rho + H(Y)_\rho &= H(X)_\sigma + H(Y)_\sigma \\
&\geq H(XY)_\sigma \\
&\geq D(\rho||\tilde{\rho}) + H(\rho). \quad (24)
\end{aligned}
$$

- We did some numerical tests, compare our bounds with Maassen and Uffink's entropic uncertainty bound, and it shows that both two bounds above become tighter when the dimension is larger than 3 and the number of the elements of POVMs both larger than 3 (When $d = 2, n = 2$ it seems that our bound and Berta's bound are not comparable).

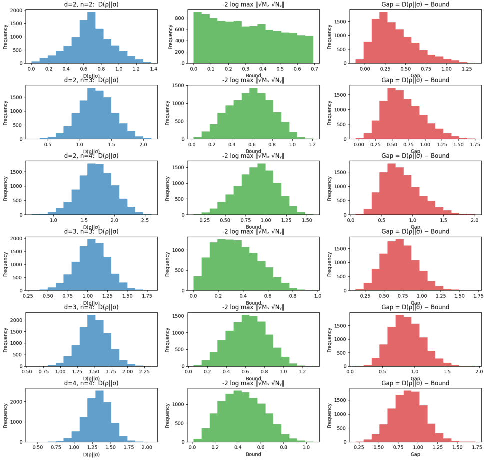

Figure: Numerical Results on Relation 2

Future Works

- The interpretation of "retrodictive uncertainty relations"
  Based on Appleby's idea [4], the "retrodictive error" is defined as the error between the initial observable system and the output measurement pointer. However, from the quantum Bayes' rule, the information that has been retrodicted is the state itself. Therefore, the retrodictive error becomes the guessing probability of the initial state.
- Interestingly, in the study of observational entropy, [5] shows that the observational entropy can reveal some information of the "recoverability" of the original state.
- Working on analytical proof of a tight bound.
  Unfortunately, the tightness of our bounds are still in numerical tests. Since the SRSR-Instrument gives the minimum disturbance on the state, it is reasonable to consider that our scenario will give a tight lower bound of "retrodictive entropic uncertainty relation".

References

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Thank You!