Testing k-block-positivity

A hierarchical SDP approach and complexity analysis

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[arXiv:2505.22100] + one paper that appears soon hopefully

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Overview

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Entanglement cones

Alice's and Bob's (original) spaces $\mathbb{C}^d$; Bipartite system $\mathbb{C}^d \otimes \mathbb{C}^d$.

- Parameterization of Schmidt rank $\leq k$ pure states: $|\psi\rangle = \sum_{i=1}^{k} |x_i\rangle \otimes |y_i\rangle$.
- Schmidt number k states = Convex linear combination of Schmidt rank $\leq k$ pure states.

Denote by SN_k the set of Schmidt number k states.
In particular, SN_1 is the set of separable states, which are convex linear combination of Schmidt rank 1 pure states.

The finer structure of entanglement – the following sequential relation:

$$\mathrm{SN}_1 \subset \mathrm{SN}_2 \subset \cdots \subset \mathrm{SN}_k \subset \cdots \subset \mathrm{SN}_d$$

Dual cones: Schmidt number witnesses

An entanglement witness is an operator $X \in \mathrm{Herm}(\mathbb{C}^d \otimes \mathbb{C}^d)$ that

- $\mathrm{Tr}(X\rho) \geq 0$ for any $\rho \in \mathrm{SN}_1$.
- $\mathrm{Tr}(X\rho) < 0$ for at least one $\rho \notin \mathrm{SN}_1$.

Generalization: (k-)block positivity. $X \in \mathrm{Herm}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is k-block-positive iff

- $\mathrm{Tr}(X\rho) \geq 0$ for any $\rho \in \mathrm{SN}_k$.

The duals of the cones of Schmidt number states, namely 1- to k-block-positivities, are presented the following sequence:

$$\mathrm{BP}_1 \supset \mathrm{BP}_2 \supset \cdots \supset \mathrm{BP}_k \supset \cdots \supset \mathrm{BP}_d$$

Note: k-block-positivity and k-positivity are related via Choi–Jamiołkowski isomorphism.

Hierarchical semidefinite programs & relaxation

- NP-hard even when $k = 1$:
  - certifying whether $\rho \in \mathrm{Pos}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is in SN_k or not;
  - certifying whether $X \in \mathrm{Herm}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is in BP_k or not.
- When $k = 1$, hierarchical semidefinite programs (SDPs) based on extendibility hierarchy (as well as Doherty-Parrilo-Spedalieri hierarchy) and quantum de Finetti theorem, provide approximative solutions via:

  Relaxation: separability $\rightarrow$ symmetric extendibility
- Exchangable extendibility hierarchy

$$\mathrm{Sep} = \mathrm{Exc}_\infty \subset \cdots \subset \mathrm{Exc}_N \subset \cdots \subset \mathrm{Exc}_2 \subset \mathrm{Exc}_1 \subset POS.$$

Bosonic extendibility hierarchy (converging faster)

$$\mathrm{Sep} = \mathrm{Ext}_\infty \subset \cdots \subset \mathrm{Ext}_N \subset \cdots \subset \mathrm{Ext}_2 \subset \mathrm{Ext}_1 \subset \mathrm{Pos}\,.$$

Separable=Infinitely extendible!

Extendibility hierarchy and quantum de Finetti theorem

Definition (Symmetric extendibility)

Exchangeable: A state $\rho_{AB} \in \mathrm{Exc}_N$ if there exists a $\rho_{AB_1\cdots B_N}$ s.t.

- $\rho_{AB_1\cdots B_N} = \pi\rho_{AB_1\cdots B_N}\pi^{-1}$ for all $\pi \in S_N$.
- $\mathrm{Tr}_{B_2\cdots B_N}(\rho_{AB_1\cdots B_N}) = \rho_{AB_1} = \rho_{AB}$.

N-Bose-extendible (N-BSE): A state $\rho \in \mathrm{Ext}_N$ is defined by requiring

- $\rho_{AB_1\cdots B_N} = \pi\rho_{AB_1\cdots B_N} = \rho_{AB_1\cdots B_N}\pi$ for all $\pi \in S_N$.

Lemma (Quantum de Finetti theorem)

If $\rho_{AB_1\ldots B_n}$ is exchangable then

$$\left\| \rho_{AB_1\cdots B_n} - \int_{U(d)} \xi_A^{\sigma} \otimes \sigma^{\otimes n} dm(\sigma) \right\|_1 \leq 4\frac{nd^2}{N}. \quad (1)$$

If $\rho_{AB_1\ldots B_n}$ is Bose-symmetric then the bound is $4\frac{nd}{N}$.

Testing k-block positivity: optimization

We will focus on testing/certifying k-block-positivity (kBP).
Many related problems, e.g., the famous *Distillability conjecture*:

$$\text{whether } (\mathbb{I} - \frac{1}{2}|\phi_4\rangle\langle\phi_4|)^{\otimes 2} \text{ is in } \mathrm{BP}_2 \text{ or not.}$$

Definition (kBP testing: optimization)

Testing kBP through solving optimization problem:

$$\min_{\rho \in \mathrm{SN}_k(d,d)} \mathrm{Tr}(X\rho). \tag{2}$$

Goals: approximately solve it via hierarchical SDPs, then answer how to:

- Reduce the SDPs by utilizing symmetries;
- Characterize the SDP complexity.

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Recast optimization problem via *k*-purification

Definition (Testing *k*-block-positivity via *k*-purification)

Let $X_k = |\phi_k\rangle\langle\phi_k| \otimes X$ where $|\phi_k\rangle = \sum_{i=1}^{k} |ii\rangle$. Then $X \in \mathrm{BP}_k$ iff $\mathscr{V}_k \geq 0$,

$$\mathscr{V}_k = \min \mathrm{Tr}(X_k \rho_k)\,,$$

$$\rho_k \geq 0\,, \quad \text{and } \mathrm{Tr}\rho_k = 1\,, \quad \text{and } \rho_k \in \underbrace{\mathrm{Sep}(\mathbb{C}^{kd} \otimes \mathbb{C}^{kd})}_{\text{separable states}} = \underbrace{\mathrm{SN}_1(\mathbb{C}^{kd} \otimes \mathbb{C}^{kd})}_{\text{states with Schmidt number 1}}\,.$$

$\mathbb{C}^{kd} \cong \mathbb{C}^k \otimes \mathbb{C}^d$ where $\mathbb{C}^k$ is the auxiliary space and $\mathbb{C}^d$ the original space.
The ρ_k has the same structure with $X_k = |\phi_k\rangle\langle\phi_k| \otimes X$,

$$\rho_k = \sum \underbrace{|i_0 i_1\rangle\langle j_0 j_1|}_{\in \mathrm{End}(\mathbb{C}^k \otimes \mathbb{C}^k)} \otimes \underbrace{\rho_{i_0 i_1; j_0 j_1}}_{\in \mathrm{End}(\mathbb{C}^d \otimes \mathbb{C}^d)}\,.$$

Indices i, j label the basis of auxiliary space, i.e., $\mathbb{C}^k = \mathrm{Span}\{|i\rangle | i = 1, \ldots, k\}$.
Remark: $\mathscr{V}_k \leq 0$ holds for any X --- *k*-block-positive iff $\mathscr{V} = 0$, otherwise.

Approaching the optimization via extendibility hierarchy

It can be approximately solved by utilizing extendibility hierarchy.

Definition (Approaching *k*-block-positivity via hierarchical SDPs)

Let $X_{k,N} = X_k \otimes \mathbb{I}_{kd}^{\otimes(N-1)}$. Then consider

$$\begin{aligned} \mathscr{S}_N &:= \min \operatorname{Tr}(X_{k,N}\rho_{k,N}) \,, \\ \rho_{k,N} &\geq 0 \,, \quad \text{and } \operatorname{Tr}\rho_{k,N} = 1 \,, \\ \rho_{k,N} &\in \operatorname{Sym}(\mathbb{C}^{kd} \otimes (\mathbb{C}^{kd})^{\otimes N}) \,. \end{aligned}$$

$\mathscr{S}_1 \leq \mathscr{S}_2 \leq \cdots \leq \mathscr{S}_N \leq \cdots \leq \mathscr{S}_\infty = \mathscr{V}_k$ due to the quantum de Finetti theorem

$$\operatorname{Sep} = \operatorname{Ext}_\infty \subset \cdots \subset \operatorname{Ext}_N \subset \cdots \subset \operatorname{Ext}_2 \subset \operatorname{Ext}_1 = \operatorname{Pos} \,.$$

where $\operatorname{Ext}_n = \operatorname{Tr}_{n-1}\operatorname{Sym}(\mathbb{C}^{kd} \otimes (\mathbb{C}^{kd})^{\otimes n})$. We write

$$\operatorname{Tr}_{N-1}\operatorname{Sym}(\mathbb{C}^{kd} \otimes (\mathbb{C}^{kd})^{\otimes N}) \overset{N\to\infty}{\longrightarrow} \operatorname{Sep}(\mathbb{C}^{kd} \otimes \mathbb{C}^{kd}) \,.$$

For Schur-Weyl: from $\bar{U} \otimes U$ to $U^{\otimes k}$

- The maximally entangled state $|\phi_k\rangle\langle\phi_k|$ carries $\bar{U} \otimes U$-symmetry.
- We are able to convert the $\bar{U} \otimes U$-symmetry to $U^{\otimes k}$-symmetry, by linear map $\mathcal{E} : \mathbb{C}^k \to (\mathbb{C}^k)^{\otimes(k-1)}$ represented by matrix

$$\mathcal{E}^{a_2 \ldots a_k}{}_i = \frac{\epsilon_{a_2 \ldots a_k i}}{\sqrt{(k-1)!}},$$

or pictorially $\mathcal{E} : \square \to \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline k-1 \\ \hline \end{array}$. The $\mathcal{E}$ never affects the entanglement.

- The maximally entangled state relates to Π_k the projector of (1^k),

$$k(\mathcal{E}^\dagger \otimes \mathbb{I}_k)\Pi_k(\mathcal{E} \otimes \mathbb{I}_k) \equiv k\mathcal{E}^\dagger \Pi_k \mathcal{E} = |\phi_k\rangle\langle\phi_k| \,.$$

- $\mathrm{U}(k)$-conjugate invariant: $\Pi_k = U^{\otimes k} \Pi_k U^{\dagger \otimes k}$ for all $U \in \mathrm{U}(k)$.

Two symmetries

Ingredients for SDP in Def. 5 at generic level N:

$$X_{k,N} := k\Pi_k \otimes \mathbb{I}_k^{\otimes(N-1)} \otimes X \otimes \mathbb{I}_d^{\otimes(N-1)} ,$$
$$\rho_{k,N} = \sum(\mathcal{E} \otimes \mathbb{I}_k^{\otimes N})|i_0 i_1 \dots i_N\rangle\langle j_0 j_1 \dots j_N|(\mathcal{E}^\dagger \otimes \mathbb{I}_k^{\otimes N}) \otimes \underbrace{\rho_{i_0 i_1 \dots i_N, j_0 j_1 \dots j_N}}_{\in \mathrm{End}(\mathbb{C}^d \otimes (\mathbb{C}^d)^{\otimes N})} .$$

SDP admits reduction from the following symmetries:

- $\mathrm{U}(k)$-symmetry carried by auxiliary systems $(\mathbb{C}^k)^{\otimes(N+k-1)}$,

$$X_{k,N} = (U^{\otimes(N+k-1)} \otimes \mathbb{I}_d^{\otimes(N+1)})X_{k,N}(U^{\dagger\otimes(N+k-1)} \otimes \mathbb{I}_d^{\otimes(N+1)}) , \ \forall U \in \mathrm{U}(k) .$$

- S_N-symmetry of Bob's $\mathbb{C}^{kd}$ because of extendibility hierarchy,

$$\rho_{k,N} = \underbrace{\Delta(\pi)}_{:=\pi\otimes\pi} \rho_{k,N} = \rho_{k,N}\Delta(\pi) , \ \forall \pi \in S_N .$$

U(k)-Reduced SDP associated with Young diagrams

Then U(k)-inv. state has the form w.r.t. λ-blocks [2025.22100]:

$$\mathcal{T}(\rho_{k,N}) = \int_{\mathrm{U}(k)} g^{\otimes(N+k-1)} \rho_{k,N} g^{\dagger\otimes(N+k-1)} dg \cong \bigoplus_{\lambda \vdash_k N+k-1} w_\lambda \rho_\lambda \,. \tag{3}$$

- We have $\min \mathrm{Tr}(X_{k,N}\rho_{k,N}) = \min_{\{\lambda \vdash_k (N+k-1)\}} \mathrm{Tr}(X_\lambda \rho_\lambda)$ from blocks

$$X_\lambda = k\Pi_k \otimes P_{\lambda^-} \otimes X \otimes \mathbb{I}_d^{\otimes(N-1)} \,,$$

where $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\lambda^- = (\lambda_1 - 1, \ldots, \lambda_k - 1)$.

Definition (Reduced SDP with trace equality)

Let $\lambda \vdash (N+k-1)$ with ρ_λ defined in Eq.(3), the reduced SDP is defined to be,

$$\mathscr{S}_\lambda := \min \mathrm{Tr}(X_\lambda \rho_\lambda) \,, \tag{4}$$

$$\text{subject to } \rho_\lambda = \Delta(\pi)\rho_\lambda = \rho_\lambda \Delta(\pi), \ \forall \pi \in S_N \,, \ \text{ and } \mathrm{Tr}\rho_\lambda = 1 \,. \tag{5}$$

Two questions

The SDP Def.5 is solved by solving over all $\lambda \vdash_k (N + k - 1)$,

$$\mathscr{S}_N = \min_{\lambda_k \vdash (N+k-1)} \mathscr{S}_\lambda . \tag{6}$$

The rest of the talk will address the following two questions:

- How to choose a family of Young diagrams (for defining hierarchy) such that we can look at fewer diagrams meanwhile make k-block-positivity testing still work?
- How to characterize the complexity such that the hierarchy collapse, the $k = d$ case, can be read from the characterization?

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Relaxing trace constraint: from equality to inequality

Definition (Reduced SDP with trace inequality)

We relax the trace constraint appeared in reduced SDP.

$$\mathscr{W}_\lambda := \min \mathrm{Tr}(X_\lambda \sigma_\lambda)\,, \tag{7}$$

$$\text{subject to } \sigma_\lambda = \Delta(\pi)\sigma_\lambda = \sigma_\lambda \Delta(\pi),\ \forall \pi \in S_N\,, \ \text{ and } \mathrm{Tr}\sigma_\lambda \leq 1\,, \tag{8}$$

If $\mathscr{S}_\lambda \geq 0$ then $\mathscr{W}_\lambda = 0$ otherwise $\mathscr{W}_\lambda < 0$.

Recall optimization problem Def.4 and note that:

- The optimization problem Def.4 is solved by pure state in the form of:

$$|\varphi\rangle = \sum (\mathcal{E} \otimes \mathbb{I}_k)|i_0 i_1\rangle \otimes (x \otimes y)|i_0 i_1\rangle\,, \ \text{ where } x, y \in M(d,k)\,. \tag{9}$$

- Let $\mu \vdash_k N$ and P_μ the central projector of μ, consider the following S_N-inv. pure state :

$$|\varphi_\mu\rangle = \sum (\mathcal{E} \otimes P_\mu)|i_0 i_1 \dots i_N\rangle \otimes (x \otimes y^{\otimes N})|i_0 i_1 \dots i_N\rangle\,. \tag{10}$$

Key theorem in Sep 1

For our purpose, just look at the situation: Set $N = kn - k + 1$ with integer n, and set $\mu = (n, (n-1)^{k-1})$, $\lambda = (n^k)$. We then look into $\mathscr{W}_{(n^k)}$ with varying n (Defining rectangular shape sequence),

$$(1^k), (2^k), \ldots, (n^k), \ldots, (\infty^k)$$

Theorem (Rectangular shape is sufficient for testing k-block-positivity)

Recall that $\mathscr{V}_k$ and $\mathscr{S}_N$ are the optimal values of optimization problem Def. 4 and SDP Def. 5 respectively. Then $\mathscr{W}_{(n^k)}$ satisfies the following bound:

$$\mathscr{S}_N \leq \mathscr{W}_{(n^k)} \leq \frac{1}{k^2}\mathscr{V}_k \leq 0\,. \tag{11}$$

This theorem implies that the sign of $\mathscr{W}_{(n^k)}$ is same as of $\mathscr{V}_k$.

Keys for the proof

- Setting $\mu = (n, (n-1)^{k-1})$, $\lambda = (n^k)$, we obtain

$$\frac{\langle\varphi_\mu|X_\lambda|\varphi_\mu\rangle}{\langle\varphi_\mu|\varphi_\mu\rangle} = \underbrace{\frac{\dim Y_{\lambda^-}}{\dim Y_\mu}}_{=\frac{n+k-1}{k(kn-k+1)}} \underbrace{\frac{\langle\phi_k|(x\otimes y)^\dagger X(x\otimes y)|\phi_k\rangle}{\mathrm{Tr}(x^\dagger x)\mathrm{Tr}(y^\dagger y)}}_{=\mathrm{Tr}(X_{k,1}|\varphi\rangle\langle\varphi|)} . \tag{12}$$

- Once $\mathscr{S}_{(n^k)} > 0$ is detected for some n, we know $\mathscr{V}_k = 0$.
- If $\mathscr{S}_{(n^k)} \leq 0$, using $\mathscr{W}_{(n^k)} = \mathscr{S}_{(n^k)}$ and $\mathscr{S}_N = \mathscr{S}_{kn-k+1} \leq \mathscr{S}_{(n^k)}$, we get the proof.

It concludes that rectangular shape is sufficient for testing k-block-positivity.
By letting $n \to \infty$, the sequential SDPs with $\mathscr{W}_{(n^k)}$ implies

$$\mathscr{S}_\infty = \mathscr{V}_k \leq \mathscr{W}_{(\infty^k)} \leq \frac{1}{k^2}\mathscr{V}_k \leq 0 .$$

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Perm-inv. Kraus operators on a Young diagram

Let $\lambda \vdash_k (N+k-1)$ be any Young diagram.

- ρ_λ in Eq.(3) could be represented by $K_{\lambda,\alpha}$ (Choi theorem),

$$\rho_\lambda = \frac{\mathbb{I}_{\mathrm{U}_\lambda^k}}{\dim \mathrm{U}_\lambda^k} \otimes \sum_{p_\lambda, p'_\lambda} |p_\lambda\rangle\langle p'_\lambda| \otimes \sum_\alpha K_{\lambda,\alpha} |p_\lambda\rangle\langle p'_\lambda| K^\dagger_{\lambda,\alpha}$$

- $K_{\lambda,\alpha}$ is exchangeable, i.e., $K_{\lambda,\alpha} = \pi K_{\lambda,\alpha} \pi^{-1}$ for all $\pi \in S_N$.
- $K_{\lambda,\alpha}$ is an intertwining map w.r.t. S_N, i.e.,

$$K_{\lambda,\alpha} \in \mathrm{Hom}_{S_N}(Y_\lambda, (\mathbb{C}^d)^{\otimes(N+1)}) \cong \mathbb{C}^d \otimes \bigoplus_{\mu \subset \lambda : \mu \searrow \lambda^-} \mathrm{U}_\mu^d ,$$

where μ come from Littlewood-Richardson.

Computational resource of reduced SDP

We relate SDP complexity to the computational resource that consists of SDP variables.

- SDP complexity (e.g., interior point method) can be characterized by the size of unconstrained positive semidefinite (PSD) matrices, which is equivalent to

  # of linear indep. Kraus operators generating PSD matrices.

- Our SDP variables are generated by intertwining maps $K_{\lambda,\alpha}$, hence, for given Young diagram λ, its SDP complexity $\mathcal{C}_\lambda$ can be characterized by

$$\mathcal{C}_\lambda = d \cdot \sum_{\mu \subset \lambda : \mu \searrow \lambda^-} \dim \mathrm{U}_\mu^d \, . \tag{13}$$

Complexity: by representation-theoretic formula

The nth-level SDP corresponds to rectangular Young diagram (n^k),

- Its μ is unique: $\mu = (n, (n-1)^{k-1})$;
- Dimension $\dim \mathbb{U}^d_{(n,(n-1)^{k-1})}$ is given by hook length formula for semistandard Young tableaux, then the complexity is,

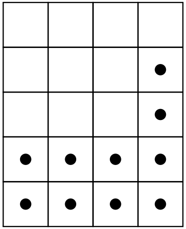
$$\mathcal{C}_{(n^k)} = d\frac{k(d+n-1)}{k+n-1}\prod_{r=1}^{k}\frac{(d+n-r-1)!(k-r)!}{(k+n-r-1)!(d-r)!}. \tag{14}$$

Two corollaries:

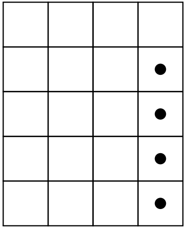
- The complexity of testing kBP has same big O as testing $(d-k)$BP.
- Hierarchy collapse (independent on n) as $k = d$ from Eq.(14).

Hierarchy collapse from representation-theoretic viewpoint

Note that $\dim \mathbb{U}^{d}_{(n,(n-1)^{k-1})}$ is equal to the dimension of the complementary Young diagram, for the example of $d = 5$ and $k = 3$ at level $n = 4$,



If $k = d = 5$, then it is easy to read from



Summary

The main results:

- We formulate a hierarchical SDP approach for testing k-block-positivity, as well as the symmetry reduction based on $\mathrm{U}(k)$- and S_N- symmetries.
- We show that the family of rectangular Young diagrams is sufficient for testing k-block-positivity.
- We obtain the characterization of SDP complexity based on the family of rectangular Young diagrams and read hierarchy collapse from the complexity formula.

Thanks for your attention!