Group Order is in QCMA

Joint work with François Le Gall and Harumichi Nishimura

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Groups

A finite group G is a finite set of elements together with a binary operation $(\cdot)$ that satisfy following group axioms:

- Closure : For all x, y in $G, (x \cdot y) \in G$.
- Associativity : For all x, y and z in G, $(x \cdot (y \cdot z)) = ((x \cdot y) \cdot z)$.
- Identity : There exists an element e in G such that, for every element a in G, the equation $(a \cdot e) = (e \cdot a) = a$ holds.
- Inverse : For each x in G, there exists an element y in G such that $(x \cdot y) = (y \cdot x) = e$.

Group Representations

- Cayley Table representation:

*	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

- Permutation Group representation: $G = \langle \pi_1, \ldots, \pi_t \rangle \leq S_n$
- Matrix Group representation: $G = \langle M_1, \ldots, M_t \rangle \leq \mathrm{GL}(d, q)$
- Black-box Group Representation

One of the most general ways to work with finite groups

□ Depending on what representation is used a computational problem can become very easy or extremely challenging.

Black-box Representation:

- $G = \langle g_1, \ldots, g_t \rangle$
- Each element of G is represented by a binary string of length $O(\log|G|)$ bits.
- We have two oracles available at unit cost:
  - String representing g and g' $\rightarrow$ [Multiplication Oracle] $\rightarrow$ String representing gg'.
  - String representing g $\rightarrow$ [Inverse Oracle] $\rightarrow$ String representing g^{-1}.

☐ In the quantum setting,

- we can feed quantum superpositions of elements to the oracles
- we assume that each element is encoded by a unique binary string (as in all prior works).

☐ Since the group is generated by $O(\log|G|)$ elements, the input size is $O(\log|G|)^2$ bits.

Group Order

Input: A group G as a black-box representation.

Question: Compute $|G|$

Group Order Verification

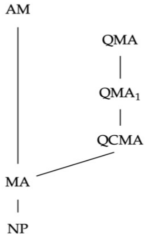
Input: A group G as a black-box representation and an integer m.

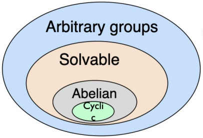
Question: Decide if $|G| = m$

- One of the most fundamental problem over groups

Prior Work

- Classical bounds:
  - Requires exponential time (in $\log|G|$), even for cyclic groups
  - Best upper bound on its complexity is the class AM. [Babai 1992]
- Quantum polynomial time algorithms:
  - Cyclic groups [Shor 1994]
  - Abelian groups [Kitaev 1995]
  - Solvable groups [Watrous 2000]





An Open Question:

What about arbitrary groups? Is it in QMA [Watrous 2000]?

Our Result: Group Order Verification is in QCMA!

Group Membership

Input: A group G as a black-box representation, $H \leq G$, and $g \in G$.

Question: Decide if $g \in H$

Group Non-Membership

Input: A group G as a black-box representation $H \leq G$, and $g \in G$.

Question: Decide if $g \notin H$

- Group Membership is in NP [Babai and Szemerédi, 1984]
- Group Non-Membership is in QMA [Watrous, 2000]
- Group Non-Membership is in QCMA under some group-theoretic assumptions [Aaronson and Kuperberg, 2007]
- **Conjecture**: Group Non-Membership is in QCMA
- Solved!
  Algorithm for Group Order gives a way to check Membership

$$g \in H \text{ if and only if } |\langle H, g\rangle| = |H|$$

Proof Strategy

Group Order Verification: A group G as a black-box representation and an integer m. Decide if $|G| = m$.

- ☐ Group Order **Divisor** Verification: Decide if m divides $|G|$.
- ☐ Group Order **Multiple** Verification: Decide if $|G|$ divides m.

Theorem 1 Group Order Divisor Verification is in QCMA.

Proof Idea:

- Let $m = p_1^{a_1} \cdots p_r^{a_r}$ be the prime decomposition of m.
- Claim: m divides $|G|$ iff G has a subgroup H_i if order $p_i^{a_i}$, for each i.
- A group of order p^a, for a prime p and integer a is solvable.
- The prover sends a set of generators for each H_i.
- The verifier checks that H_i solvable, and then check if $|H_i| = p_i^{a_i}$ using Watrous' algorithm for solvable groups.

Theorem 2 Group Order Multiple Verification is in QCMA.

⋆ Remaining part of the talk.

Checking that $|G|$ divides m: Strategy

- A group with no nontrivial normal subgroup is called Simple group.
- A composition series of G is a list of subgroups $H_0, H_1, \dots, H_s$ for some integer s, such that
  - $\{e\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_s = G$;
  - the quotient group H_i/H_{i-1} is **simple group** for each $i \in [s]$.
- Each group has a composition series but it is unknown how to compute it efficiently.
- $|H_0| \cdot |H_1/H_0| \cdot |H_2/H_1| \cdots |H_s/H_{s-1}| = |G|$

This suggests a strategy to compute $|G|$

(i) ask the prover to send a composition series

(ii) check that each H_i/H_{i-1} is simple and compute its order

Remark: The "classification theorem of finite simple groups" (about 15,000-page long proof) states that every finite simple group belongs to one of 18 infinite families of simple groups, or is one of 26 sporadic simple groups. As a consequence, each simple group can be described by a short string called its standard name (its order can be easily obtained from its standard name).

(i) Ask the prover to send a composition series and the standard name w_i of H_i/H_{i-1}

(ii) Check: H_i/H_{i-1} is isomorphic to the simple group with standard name w_i

- For the 26 sporadic simple groups this is trivial (since they have constant order)
- For 17 of the infinite families, this can be done in classical polynomial time using a classical witness using the presentation-test (witness: a short presentation of the simple group in terms of generators and relations)
- We do not know how to do it for the "Ree groups", since it is unknown if Ree groups have a short presentation

Use a randomized homomorphism test.

- The family Ree groups of rank 1 indexed by a positive integer a. Write $q = 3^{2a+1}$. The Ree group of rank one, which we denote by $\mathrm{R}(q)$, is the subgroup of $\mathrm{GL}(7, q)$ generated by the following three matrices:

$$\Gamma_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \Gamma_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma_3 = \begin{bmatrix} \omega^t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega^{1-t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^{2t-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^{1-2t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^{t-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega^{-t} \end{bmatrix},$$

where $t = 3^a$ and ω be a primitive element of $\mathbb{F}_q$. The order of $\mathrm{R}(q)$ is $q^3(q^3 + 1)(q - 1)$.

Second Trial: Randomized Homomorphism Test for the Ree Group

Goal: check if a group Σ is isomorphic to a known group S (think of $\Sigma = H_i/H_{i-1}$ and $S = \langle s_1, \ldots, s_k \rangle = \mathrm{R}(q)$).

- We ask Prover to send elements $g_1, \ldots, g_k \in \Sigma$.
- If $\Sigma \cong S$ and Prover is honest, he will send $g_i = \phi(s_i)$ for each $i \in [k]$, for some isomorphism $\phi \colon S \to \Sigma$.
- For the checking procedure, Verifier defines a map $f : S \to \Sigma$ by extending the partial map $s_i \mapsto g_i$ into a map on all S as if it were a homomorphism.
- For instance, for an element $s \in S$ that can be written as $s = s_1 s_2 s_1 s_3$, Verifier will set $f(s) = g_1 g_2 g_1 g_3$.
- Verifier takes two elements s and s' uniformly at random in S and checks if

$$f(ss') = f(s)f(s') \iff f(ss')f(s')^{-1}f(s)^{-1} \in H_{i-1} \qquad (1)$$

To be successful, this approach has to satisfy three important requirements:

A. Verifier needs to be able to efficiently represent an arbitrary element $s \in S$ as a product of elements from the fixed set $\{s_1, \ldots, s_k\}$. This representation should also be unique for f to be well-defined.
   - For $S = \mathrm{R}(q)$ ([Babai, Beals, Seress 2009]+ quantum algorithms)

B. Verifier needs to be able to efficiently check that the homomorphism is actually an isomorphism, i.e., a bijection.
   - "easy" for S = R(q) (since a simple group has no nontrivial normal subgroup)

C. Verifier needs to be able to efficiently check if $f(ss')f(s')^{-1}f(s)^{-1} \in H_{i-1}$ holds.
   - Seems hard for arbitrary H_{i-1}, but Membership testing in Solvable group is in BQP [Watrous, 2000]

(Modified) Babai-Beals filtration [Babai and Beals, 1999]:

For any group G, there exists a solvable subgroup H_0 and elements $\beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_s \in G$ such that when defining $H_i = \langle H_0, \beta_1, \gamma_1, \ldots, \beta_i, \gamma_i \rangle$ for each $i \in [s]$ we have

$$\{e\} \trianglelefteq H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_s \trianglelefteq \mathrm{Pker}(G) \trianglelefteq G$$

- $G/\mathrm{Pker}(G) \leq Sym(s)$
- $\mathrm{Pker}(G)/H_s$ solvable;
- Each H_i/H_{i-1} is simple;
- $H_i/H_{i-1} \cong \langle H_0, \beta_i, \gamma_i \rangle / H_0$, for all $i \in [s]$

Group Order Verification

Input: A group G as a black-box representation and an integer m.

Question: Decide if $|G| = m$

Open Problem [Watrous, 2000]
Is Group Order Verification in QMA? SOLVED!

Group Non-Membership

Input: A group G as a black-box representation, $H \leq G$, and $g \in G$.

Question: Decide if $g \notin H$

Conjecture [Aaronson and Kuperberg, 2007]
Group Non-Membership is in QCMA. SOLVED!

Thank you! Questions?