The Hopf Superalgebra of Two-Colored Graphs

joint work with Hattori Masamune and Shintaro Yanagida

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Contents

Introduction

k : field, $G = (V, E)$: simple graph, $[G]$: equivalent class of G
$\mathcal{G} := Span_k \langle [G] | G : simple\ graph \rangle$

$$\Delta([G]) := \sum_{\substack{V_1, V_2 \\ V_1 \sqcup V_2 = V}} [G|_{V_1}] \otimes [G|_{V_2}],$$
$$[G] \cdot [H] := [G \sqcup H],$$
$$u : k \to \mathcal{G}; \gamma \mapsto \gamma[\emptyset],$$
$$\varepsilon : \mathcal{G} \mapsto k; [G] \mapsto \begin{cases} 1 & \text{if } G = \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

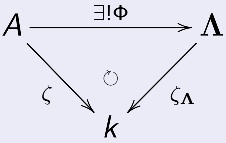
$\mathcal{G}$ becomes Hopf algebra, it is called **chromatic Hopf algebra**[1].

[1] W. R. Schmitt, *Hopf algebras of combinatorial structures*, Can. J. Math., **45** (1993), 412–428.

Introduction

Fact. [Grinberg, Reiner : Remark 7.1.4.][2]

k : field, A : cocommutative connected graded Hopf algebra, $\zeta : A \to k$ (character),
$\mathbf{\Lambda}$: ring of symmetric functions, $\zeta_{\mathbf{\Lambda}} : \mathbf{\Lambda} \to k; f(x_1, \cdots, x_n) \mapsto f(1, 0, \cdots, 0)$.
Then there uniquely exists morphism of graded Hopf algebra Φ such that



[2] D. Grinberg, V. Reiner, *Hopf Algebras in Combinatorics*, expository notes (2020), 282 pp., arXiv:1409.8356v7.

Introduction

- <u>cocomm</u> $\tau : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}; [G] \otimes [H] \to [H] \otimes [G]$, then $\tau \circ \Delta([G]) = \Delta([G])$.
- <u>grading</u> $\mathcal{G}_n := \{[G]; G = (V, E) : \text{simple graph}, \#V = n\}$, then $\mathcal{G} = \bigoplus_{n \geq 0} \mathcal{G}_n$.
- <u>connectivity</u> $\mathcal{G} = k[\emptyset] = k$.

Therefore there exists a morphism of graded Hopf algebra $\Phi : \mathcal{G} \to \mathbf{\Lambda}$. The image $\Phi(\mathcal{G})$ is called **chromatic symmetric functions**[3].

Question

Can we make chromatic symmetric functions in superspace ?

[3] R. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*, Adv. Math., **111** (1995), 166–194.

Superspace and Hopf superalgebra

Definition (superspace)

Vector space V is called **superspace** if V is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space.

Notation : $V = V_0 \oplus V_1$, V_0 : even part, V_1 : odd part.

Definition (parity of linear map)

V, W : superspace. Linear map $f : V \to V$ is **even** (resp. **odd**) if $f(V_0) \subset W_0$, $f(V_1) \subset W_1$ (resp. $f(V_0) \subset W_1$, $f(V_1) \subset W_0$)

Some properties

V, W : superspace, f, g, h, i : linear map between superspace. Then

$$(V \otimes W)_0 = V_0 \otimes W + V_1 \otimes W_1,$$
$$(V \otimes W)_1 = V_0 \otimes W_1 + V_1 \otimes W_0,$$
$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w),$$
$$(f \otimes g) \circ (h \otimes i) = (-1)^{|g||h|} f \circ h \otimes g \circ i.$$

Definition (Hopf superalgebra)

Hopf superalgebra is a Hopf object in the category of superspace.

Symmetric functions in superspace

x_i : even variable, θ_i : odd variable (i.e. $\theta_i\theta_j = -\theta_j\theta_i$)
We call $(x_1, \dots, x_N, \theta_1, \dots, \theta_N)$ supervariable and for all $\sigma \in \mathfrak{S}_n$ we define

$$\sigma(f(x_1, \dots, x_N, \theta_1, \dots, \theta_N)) := f(x_{\sigma(1)}, \dots, x_{\sigma(N)}, \theta_{\sigma(1)}, \dots, \theta_{\sigma(N)})$$

$k[x,\theta]_N^{\mathfrak{S}_N}$: ring of symmetric polynomials in N supervariables.
$(k[x,\theta]_N^{\mathfrak{S}_N})_{n,m}$: homogeneous elements with $\deg_x = n$, $\deg_\theta = m$.

$$\rightsquigarrow k[x,\theta]_N^{\mathfrak{S}_N} = \bigoplus_{\substack{n\geq 0 \\ 0\leq m\leq N}} (k[x,\theta]_N^{\mathfrak{S}_N})_{n,m}$$

Notation : $\mathbf{\Lambda}_{n,m} := \varprojlim_N (k[x,\theta]_N^{\mathfrak{S}_N})_{n,m}$, $\boxed{\mathbf{\Lambda} := \bigoplus_{n,m\geq 0} \mathbf{\Lambda}_{n,m}}$.

Definition (super partition)

$\Lambda = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_N)$, $\Lambda_i \in \mathbb{Z}_{\geq 0}$ $(1 \leq i \leq m), \Lambda_i \in \mathbb{Z}_{>0}$ $(m+1 \leq i \leq N)$ $(\Lambda_1, \ldots, \Lambda_m)$ and $(\Lambda_{m+1}, \ldots, \Lambda_N)$ are partitions. We call m **fermion degree** of Λ and $|\Lambda| := \sum_{1 \leq i \leq N} \Lambda_i$ **total degree**.

For super partition $(\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_N)$ we define

$$m_\Lambda(x, \theta) = \sum_{\sigma \in \mathfrak{S}_N} \theta_{\sigma(1)} \cdots \theta_{\sigma(m)} x_{\sigma(1)}^{\Lambda_1} \cdots x_{\sigma(N)}^{\Lambda_N}$$

Example

$$m_{(0;1,2)}(x, \theta) = \theta_1 x_2 x_3^2 + \theta_2 x_1 x_3^2 + \theta_3 x_2 x_1^2 + \theta_1 x_3 x_2^2 + \theta_2 x_3 x_1^2 + \theta_3 x_1 x_2^2$$

$$m_{(1;1,2)}(x, \theta) = \theta_1 x_1 x_2 x_3^2 + \theta_2 x_1 x_2 x_3^2 + \theta_3 x_2 x_3 x_1^2 + \theta_1 x_1 x_3 x_2^2 + \theta_2 x_2 x_3 x_1^2 + \theta_3 x_1 x_3 x_2^2$$

Combinatorial Hopf superalgebra

Definition (super character)

ϵ : odd element, k : field, $k[\epsilon] := k + k\epsilon$, H : Hopf superalgebra
$\zeta : H \to k[\epsilon]$ is a **super character** if ζ is even and $\zeta(ab) = \zeta(a)\zeta(b)$ for all $a, b \in H$.

We call a pair (H, ζ) combinatorial Hopf superalgebra.

Example of super character

$$\zeta_S : \mathbf{\Lambda} \to k[\epsilon]; f(x_1, \cdots, x_N, \theta_1, \cdots, \theta_N) \mapsto f(1, 0, \cdots, 0, \epsilon, 0, \cdots, 0)$$

When H is a connected graded Hopf superalgebra we call $(\mathbf{\Lambda}, \zeta_S)$ a combinatorial Hopf superalgebra.

Proposition [Hattori, Y., Yanagida 2025]

Let (A, ζ) be a cocommutative combinatorial Hopf superalgebla. Then there uniquely exists a morphism of Hopf superalgebra $\Psi : A \to \mathbf{\Lambda}$ such that

$$\begin{array}{ccc} A & \xrightarrow{\exists! \Psi} & \mathbf{\Lambda} \\ & \searrow^{\zeta} \quad \circlearrowleft \quad \swarrow_{\zeta_S} & \\ & k[\epsilon] & \end{array}$$

Moreover,

$$\Psi(a) = \sum_{\Lambda:\text{super partition } s.t. |\Lambda| = k} \zeta_\Lambda(a) m_\Lambda \qquad \forall a \in A^k$$

For superpartition $(\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_N)$ the map ζ_Λ is the composition

$$\zeta_\Lambda : A^k \xrightarrow{\Delta^{(N-1)}} A^{\otimes N} \xrightarrow{proj} A_1^{\Lambda_1} \otimes \cdots \otimes A_1^{\Lambda_m} \otimes A_0^{\Lambda_{m+1}} \otimes \cdots \otimes A_N^{\Lambda_N} \xrightarrow{\zeta^{\otimes N}} k$$

We write $\zeta = \zeta_0 + \epsilon\zeta_1$ and $\zeta(A) = \zeta_0(A_0) + \epsilon\zeta_1(A_1)$.
(e.g. $(\zeta \otimes \zeta)(A_1^{\Lambda_1} \otimes A_0^{\Lambda_2}) = \zeta_1(A_1^{\Lambda_1})\zeta_0(A_0^{\Lambda_2}) \in k$)

Remark

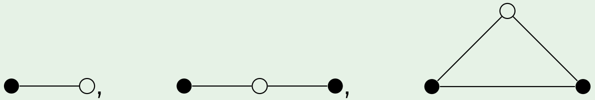
This proposition means that (Λ, ζ_S) is the terminal object of the category of cocommutative combinatorial Hopf super algebras. More generally, quasi-symmetric functions in superspace, sQSym, is the terminal object of the category of combinatorial Hopf superalgebra.

Chromatic Polynomials in superspace

Definition (two-colored graph)

For a finite simple graph $G = (V, E)$ we take a subset $W \subset V$. Then we call (V, W, E) **two-colored graph**. $V \backslash W$: black vertex, W : white vertex.

Example



Notations

$$\mathcal{G}' := \mathrm{Span}_k \langle [G] : \text{G is a finite simple two-colored graph} \rangle ,$$
$$\mathcal{C} := \left\langle [G_1|G_2] - (-1)^{\#W_1 \#W_2}[G_2|G_1] : G_1, G_2 \text{ are connected two-colored graph} \right\rangle_{\mathrm{ideal}} ,$$
$$\mathcal{G} := \mathcal{G}'/\mathcal{C}.$$

We can introduce bi-grading to $\mathcal{G}$ by

$$\mathcal{G}_{m,n} := \mathrm{Span}_k \langle [G] : G = (V, W, E), \#V = m, \#W = n \rangle .$$

Then $\mathcal{G} = \bigoplus_{m,n \geq 0} \mathcal{G}_{m,n}$, $\mathcal{G}_{0,0} = k[\emptyset] = k$.

Hopf superalgebra structure of $\mathcal{G}$

Hopf superalgebra structure

$$[G_1] \cdot [G_2] := [G_1 | G_2], \quad \text{(disjoint union of graphs)}$$

$$\Delta([G]) := \frac{1}{2} \sum_{\substack{V_1, V_2 \\ V_1 \sqcup V_2 = V}} \{[G|_{V_1}] \otimes [G|_{V_2}] + (-1)^{\# W_1 \# W_2} [G|_{V_2}] \otimes [G|_{V_1}]\}$$

$$u : k \to \mathcal{G}; \gamma \mapsto \gamma[\emptyset],$$

$$\varepsilon : \mathcal{G} \mapsto k; [G] \mapsto \begin{cases} 1 & \text{if } G = \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

$\rightsquigarrow$ $\mathcal{G}$ is a connected cocommutative graded Hopf superalgebra.

Examples

$$
\begin{aligned}
\Delta([\bullet\!-\!\circ]) &= [\bullet\!-\!\circ] \otimes 1 + [\bullet] \otimes [\circ] + [\circ] \otimes [\bullet] + 1 \otimes [\bullet\!-\!\circ]. \\
\Delta([\circ\!-\!\circ]) &= [\circ\!-\!\circ] \otimes 1 + [\circ] \otimes [\circ] + [\circ] \otimes [\circ] + 1 \otimes [\circ\!-\!\circ] \\
&= [\circ\!-\!\circ] \otimes 1 + 1 \otimes [\circ\!-\!\circ], \\
\Delta([\overset{\circ}{\bullet\!-\!\bullet}]) &= [\overset{\circ}{\bullet\!-\!\bullet}] \otimes 1 + 1 \otimes [\overset{\circ}{\bullet\!-\!\bullet}] + 2[\bullet] \otimes [\bullet\!-\!\circ] \\
&\quad + 2[\bullet\!-\!\circ] \otimes [\bullet] + [\bullet\!-\!\bullet] \otimes [\circ] + [\circ] \otimes [\bullet\!-\!\bullet].
\end{aligned}
$$

Remark : $1 = [\emptyset]$

super character of $\mathcal{G}$

$$\zeta_{\mathrm{ch}}([G]) := \begin{cases} 1 & (E = \emptyset, \#W = 0), \\ \varepsilon & (E = \emptyset, \#W = 1), \\ 0 & (\text{otherwise}) \end{cases}$$

$\rightsquigarrow$ $\boxed{(\mathcal{G}, \zeta_{\mathrm{ch}}) \text{ is a cocommutative combinatorial Hopf superalgebra.}}$

$\rightsquigarrow$ $\exists! \Psi : \mathcal{G} \to \mathbf{\Lambda}$ (graded even map).

We call $\Psi(\mathcal{G})$ **chromatic symmetric function in superspace**.

Example

$$\Psi([\bullet]) = m_{(\emptyset;1)}(x, \theta),$$

$$\Psi([\bullet\!\!-\!\!\circ]) = m_{(0;1)}(x, \theta),$$

$$\Psi\Big(\Big[\begin{smallmatrix} & \circ & \\ \bullet & \!\!-\!\! & \bullet \end{smallmatrix}\Big]\Big) = 2m_{(0;1,1)}(x, \theta).$$

Proposition [Hattori, Y., Yanagida 2025]

Let $K_{n+1,1}$ be a complete graph such that $\#V = n+1$, $\#W = 1$. Then

$$\Psi([K_{n+1,1}]) = n! m_{(0;1^n)}.$$

Remark

When $\#W \geq 2$, we have $\Psi([G]) = 0$

Definition (proper coloring)

$G = (V, W, E)$
$f : V \to \mathbb{N}$ is a **proper coloring** if $f(u) \neq f(v)$ ($u, v \in V$ such that $\{u, v\} \in E$)

Proposition [Hattori, Y., Yanagida 2025]

Let $G = (V, W, E)$ be a two-colored graph. Then

$$\Psi([G]) = \sum_{f\text{:proper coloring}} (x, \theta)_f$$

where $(x, \theta)_f := \prod_{w \in W} \theta_{f(w)} \prod_{v \in V \setminus W} x_{f(v)}$

Future

- Can we make LLT polynomials in superspace?
- What is super analogue of chromatic quasi-symmetric functions?
- Are there some relations between affine Hecke algebras and chromatic quasi-symmetric functions in superspace? (What is a super analogue of (q, t)-chromatic symmetric polynomials ?)
- Can we make super analogue of relation between chromatic quasi-symmetric functions and character of $GL_n(\mathbb{F}_q)$?