Quantum Retrodiction from First Principles and Its Applications

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A detector clicks. What can we infer about its input?

A classical detector reports x with probability $P(x|i)$ when its input is i. Given a prior $\gamma(i)$, the predicted probability of the click is $\pi(x) = \sum_i P(x|i)\gamma(i)$.

Upon observing outcome x

$$R_P^\gamma(i|x) = \frac{\overbrace{\gamma(i)}^{\text{prior}}\ \overbrace{P(x|i)}^{\text{likelihood}}}{\pi(x)}, \qquad \pi(x) > 0.$$

Averaging these posteriors with weights $\pi(x)$ gives back γ. But the rule that ignores every click and always returns γ also does this. What else singles out Bayesian retrodiction?

Consistency of retrodiction

For every channel Φ and prior γ, a single rule must assign a reverse channel R^{γ}_{Φ} taking $\tau = \Phi(\gamma)$ back to γ.

- Sequential composition. Reverse the stages in the opposite order, $R^{\gamma}_{\Psi\circ\Phi} = R^{\gamma}_{\Phi} \circ R^{\tau}_{\Psi}$.
- Parallel composition. For independent processes with product prior, $R^{\gamma\otimes\eta}_{\Phi\otimes\Psi} = R^{\gamma}_{\Phi} \otimes R^{\eta}_{\Psi}$.
- Inversion of reversible channels. $R^{\gamma}_{\Phi} = \Phi^{-1}$ for permutations in classical theory and unitary channels in quantum theory.
- Involutivity. Applying the rule a second time, now with prior τ, returns the original channel, $R^{\tau}_{R^{\gamma}_{\Phi}} = \Phi$.

A. J. Parzygnat and F. Buscemi, Quantum **7**, 1013 (2023)

Bayes and Petz from consistency of retrodiction

Classical systems. The first three requirements already determine Bayes' inverse. Involutivity follows.

Quantum systems. The first three requirements still allow rotations that preserve the reference states. Involutivity eliminates this freedom.

Uniqueness of quantum retrodiction

The four requirements uniquely determine the Petz transpose map

$$R^{\gamma}_{\Phi}(X) = \sqrt{\gamma}\,\Phi^{\dagger}\left(\tau^{-1/2} X \tau^{-1/2}\right)\sqrt{\gamma}, \qquad \tau = \Phi(\gamma).$$

Quantum channels act between finite-dimensional full matrix algebras on the reference supports.

J. Surace, C. Li, and F. Buscemi, arXiv:2610.40158

Two derivations of the same measurement retrodiction

Let $\mathcal{M}$ be the quantum-to-classical channel associated with a POVM $\{M_x\}$, and let γ be a prior with $\pi(x) = \mathrm{Tr}[M_x\gamma] > 0$ for every outcome x.

The structural axioms assign to $\mathcal{M}$ its Petz transpose $\mathcal{P}_{\mathcal{M},\gamma}$. Independently, minimum change asks which reverse channel makes the reverse and forward joint states as close as possible. For a broad class of divergences and any full-support outcome distribution, the *unique minimizer* is the same Petz map.

J. Kuang, K. Torii, and F. Buscemi, QST **11**, 025049 (2026)

Upon observing outcome x

Both arguments give the same retrodicted input state,

$$\gamma_x = \mathcal{P}_{\mathcal{M},\gamma}(|x\rangle\langle x|) = \boxed{\frac{\sqrt{\gamma}\, M_x \sqrt{\gamma}}{\pi(x)}}.$$

Reading the correspondence in the other direction

The reverse of a measurement prepares a state for each outcome. By involutivity, we can also start from a preparation and read off its measurement.

Suppose a device prepares τ_i on input i, and probabilities $g_i > 0$ give the average $\gamma = \sum_i g_i\tau_i$. Solving the measurement update for its effect gives

$$\tau_i = \frac{\sqrt{\gamma}\, E_i \sqrt{\gamma}}{g_i} \qquad \Longleftrightarrow \qquad E_i = g_i\, \gamma^{-1/2}\tau_i\gamma^{-1/2}.$$

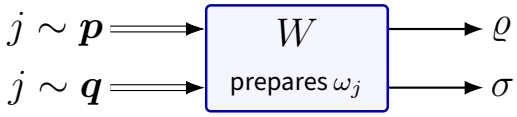
Indeed, $\sum_i E_i = I$ on the support of γ.

What happens if we use the measurement obtained from one preparation to test *another* preparation?

Can we compare two quantum states by how they are prepared?

Matsumoto's reverse test

On input j, the device W prepares ω_j. Choosing j according to p or q gives the average state ϱ or σ, respectively.



$$\varrho = W(p) = \sum_j p_j \omega_j, \qquad \sigma = W(q) = \sum_j q_j \omega_j.$$

Minimum over reverse tests

For faithful ϱ, σ and operator-convex f with $f(1) = 0$, the least classical divergence over all reverse tests is the maximal quantum f-divergence,

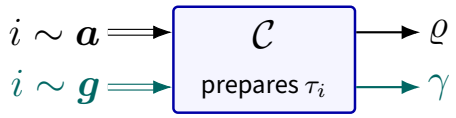
$$\min_{W,p,q:\, W(p)=\varrho,\, W(q)=\sigma} D_f(q\|p) = \mathrm{Tr}\left[\varrho\, f\left(\varrho^{-1/2}\sigma\varrho^{-1/2}\right)\right] = \widehat{D}_f(\sigma\|\varrho).$$

K. Matsumoto, arXiv:1311.4722

Two separate reverse tests with a common reference

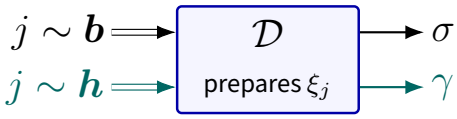
Choose a faithful reference state γ. Use a Matsumoto reverse test for each pair, with positive reference laws g and h on the respective label sets.

Pair (ϱ, γ)



$$\min D_f(a\|g) = \widehat{D}_f(\varrho\|\gamma)$$

Pair (σ, γ)



$$\min D_f(b\|h) = \widehat{D}_f(\sigma\|\gamma)$$

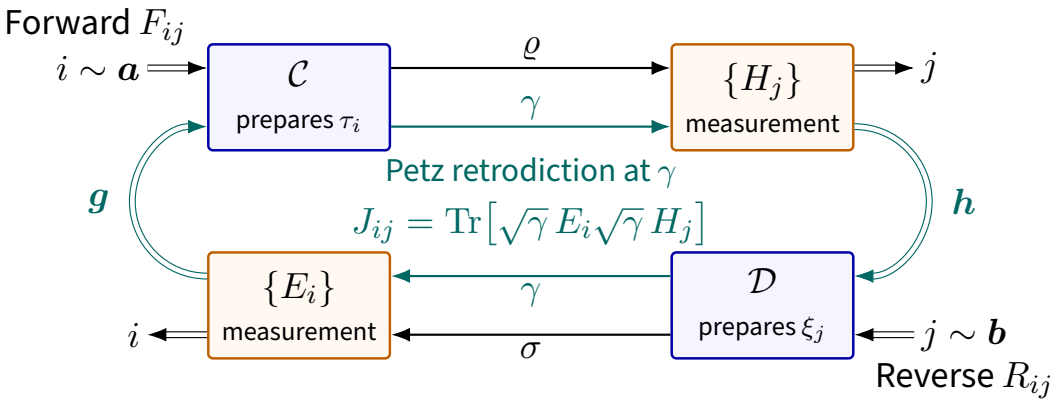
Each test compares one state with γ. To compare the preparations of ϱ and σ directly, we still need an experiment that couples the label i of the first device to the label j of the second.

Coupling two reverse tests by Bayes–Petz retrodiction

Reverse each preparation at its reference law. The corresponding Petz measurements are

$$E_i = g_i\, \gamma^{-1/2} \tau_i \gamma^{-1/2}, \qquad H_j = h_j\, \gamma^{-1/2} \xi_j \gamma^{-1/2}.$$

Use the measurement from each device to test the output of the other.



$$F_{ij} = a_i \,\mathrm{Tr}[\tau_i H_j]\,, \qquad R_{ij} = b_j \,\mathrm{Tr}[\xi_j E_i]\,.$$

Both experiments now give a record on the same alphabet of pairs (i, j).

Which preparations minimize the classical divergence?

Fix faithful states ϱ, σ, γ. We minimize the classical divergence $D_f(R\|F) = \sum_{i,j} F_{ij} f(R_{ij}/F_{ij})$ over all admissible implementations. Write

$$A = \gamma^{-1/2} \varrho\, \gamma^{-1/2} = \sum_i \lambda_i P_i, \qquad B = \gamma^{-1/2} \sigma\, \gamma^{-1/2} = \sum_j \mu_j Q_j.$$

Spectral optimality

Preparations with Petz measurements $E_i = P_i$ and $H_j = Q_j$ attain the minimum for every operator-convex f with $f(1) = 0$. Its value is the Petz quasi-entropy

$$\min_{\text{admissible}} D_f(R\|F) = S_f^{\sqrt{\gamma}}(B\|A) = \operatorname{Tr}\left[Y^\dagger f(\Delta_{B|A})(Y)\right]$$
$$= \sum_{i,j} \lambda_i f\left(\frac{\mu_j}{\lambda_i}\right) \operatorname{Tr}[\sqrt{\gamma}\, P_i \sqrt{\gamma}\, Q_j]\,,$$

where $Y = \sqrt{\gamma}\, A^{1/2}$ and $\Delta_{B|A}(X) = BXA^{-1}$.

F. Buscemi, arXiv:2610.29696

Matsumoto's test as a special case

Choose $\gamma = \varrho$

Since $A = I$, one label $*$ suffices for $\mathcal{C}$. It prepares ϱ and its reverse has the single effect I. Hence

$$F_{*j} = \operatorname{Tr}[\varrho H_j] = h_j, \qquad R_{*j} = b_j.$$

The same device $\mathcal{D}$ prepares ϱ from h and σ from b, exactly as in Matsumoto's test. Either endpoint $\gamma \in \{\varrho, \sigma\}$ gives the maximal quantum f-divergence.

Choose $\gamma = I/d$

The spectral measurements of ϱ and σ give the Nussbaum–Szkoła distributions and hence the standard quantum f-divergence.

For $f(t) = t \log t$, the two cases give the Belavkin–Staszewski and Umegaki relative entropies, respectively.

Reference choices and optimized values for retrodictive tests

Classical optimization	**Reference γ**	**Quantum value**
$\min D_f(R\|F)$	Any fixed $\gamma > 0$	Petz quasi-entropy $S_f^{\sqrt{\gamma}}(B\|A)$
	I/d	Standard $D_f^{\mathrm{Petz}}(\sigma\|\varrho)$
	$\gamma_t, \quad 0 < t < 1$	Geodesic $D_f^t(\sigma\|\varrho)$
	ϱ or σ	Maximal $\widehat{D}_f(\sigma\|\varrho)$
$\max \sum_{i,j} \sqrt{F_{ij} R_{ij}}$	I/d	Holevo fidelity
	ϱ or σ	Matsumoto fidelity
	γ_{-1}	Uhlmann fidelity (root convention)
$\min_{\substack{\gamma>0 \\ [\gamma,\sigma]=0}} \min D_\beta(F\|R)$	$\gamma_{1-1/\beta}$	Sandwiched Rényi divergence $\widetilde{D}_\beta(\varrho\|\sigma)$ $\beta \in (0,1) \cup (1,2]$

Here $A = \gamma^{-1/2} \varrho \gamma^{-1/2}$, $B = \gamma^{-1/2} \sigma \gamma^{-1/2}$, and $\gamma_t = \sigma^t / \operatorname{Tr}[\sigma^t]$. All references are faithful and all implementations are finite and admissible.

Thank you.

The spectral value of the minimum

Write the spectral decompositions $A = \sum_i \lambda_i P_i$ and $B = \sum_j \mu_j Q_j$, with $A = \gamma^{-1/2} \varrho \gamma^{-1/2}$ and $B = \gamma^{-1/2} \sigma \gamma^{-1/2}$. The common reference gives the joint weights

$$J_{ij} = \operatorname{Tr}[\sqrt{\gamma}\, P_i \sqrt{\gamma}\, Q_j] \,.$$

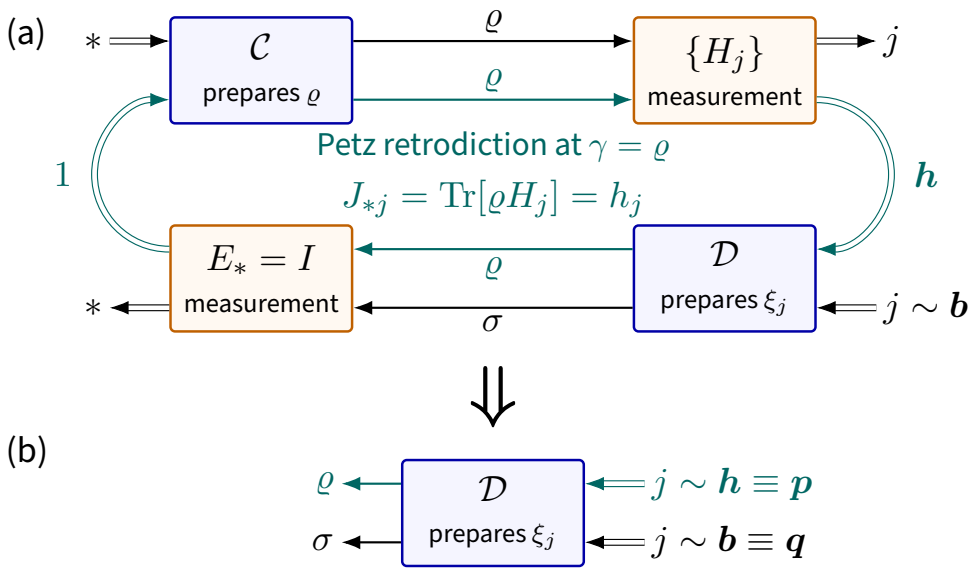
At the optimum $F_{ij} = \lambda_i J_{ij}$ and $R_{ij} = \mu_j J_{ij}$. Thus

$$\boxed{\min_{\text{admissible}} D_f(R\|F) = \sum_{i,j} J_{ij}\, \lambda_i\, f\left(\frac{\mu_j}{\lambda_i}\right).}$$

At $\gamma = I/d$, $J_{ij} = \operatorname{Tr}[P_i Q_j]/d$, which gives the overlap weights of the Nussbaum–Szkoła distributions.

The reduction at $\gamma = \varrho$

Choose $\mathcal{C}(*) = \varrho$, so $E_* = I$. Suppressing the deterministic label $*$ leaves the two uses of the same preparation device $\mathcal{D}$.



Other approaches to Bayesian conditioning

$$\underbrace{\Pr(i|x_o)}_{\text{posterior}} = \frac{\overbrace{\Pr(i)}^{\text{prior}}\ \overbrace{\Pr(x_o|i)}^{\text{likelihood}}}{\underbrace{\Pr(x_o)}_{\text{predicted probability}}} \quad \propto \overbrace{\Pr(i)}^{\text{prior}}\ \overbrace{\Pr(x_o|i)}^{\text{likelihood}}$$

Historical approaches include the following.

- *Foundational programs* of de Finetti, (Harold) Jeffreys, Savage, and Cox
- *Probability kinematics* by (Richard) Jeffrey
- *Virtual evidence method* by Judea Pearl
- *Minimum change principle* by Diaconis and Zabell

Works this talk is based on

- F. Buscemi and V. Scarani, *Fluctuation theorems from Bayesian retrodiction*. Physical Review E, vol. 103, 052111 (2021).
- A. J. Parzygnat and F. Buscemi, *Axioms for retrodiction: achieving time-reversal symmetry with a prior*. Quantum, vol. 7, 1013 (2023).
- G. Bai, F. Buscemi, and V. Scarani, *Quantum Bayes' rule and Petz transpose map from the minimum change principle*. Physical Review Letters 135, 090203 (2025).
- J. Kuang, K. Torii, and F. Buscemi, *Quantum measurement retrodiction and entropic uncertainty relations*. Quantum Science and Technology 11, 025049 (2026).
- F. Buscemi, *Witnessing Quantum Bayesian Inference beyond Classical Learning*. Preprint arXiv:2610.09293 (2026).
- F. Buscemi, *Quantum f-divergences from retrodictive tests: a unifying classical variational principle*. Preprint arXiv:2610.29696 (2026).
- J. Surace, C. Li, and F. Buscemi, *Inevitability of the Petz transpose map*. Preprint arXiv:2610.40158 (2026).

Backup