Categorical Foundations of Topological Orders

Condensation Completion and Symmetry Enrichment

Gen Yue

Department of Physics
The Chinese University of Hong Kong

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Main references

G. Yue, L. Wang and T. Lan, *Condensation Completion and Defects in 2+1D Topological Orders*. Phys. Rev. B **112**, 104106 (2025).

T. Lan, G. Yue and L. Wang, *Category of SET orders*. JHEP **11** (2024) 111.

D. Gaiotto and T. Johnson-Freyd, *Condensations in higher categories*. arXiv:1905.09566.

T. Johnson-Freyd, *On the Classification of Topological Orders*. Commun. Math. Phys. **393**, 989–1033 (2022).

L. Kong and H. Zheng, *Categories of quantum liquids I*. JHEP **08** (2022) 070.

L. Kong and H. Zheng, *Categories of quantum liquids II*. Commun. Math. Phys. **405**, 203 (2024).

Physical systems with symmetry are representations

Representation principle

Physical systems with a given symmetry form **representation categories**.

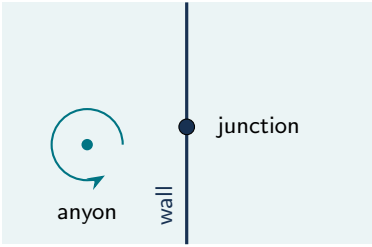
To make the representation principle precise for particular systems, we need:

- an appropriate description of **symmetry**
- an appropriate **target** that includes complete enough physical data

In this talk, we will focus on **topological orders**, with **finite generalized symmetries**.

Topological Orders and Their Macroscopic Descriptions

- Topological orders are strongly correlated **gapped quantum liquid** phases.
- The topological defects in a topological order in the long wave length limit have rather rich algebraic structures, and are described by (higher) **category theory**.



(2+1)D Topological Orders

Consider a bosonic $(2+1)$D topological order in the finite semisimple setting.

Anyon excitations

The **fusion and braiding** of point-like excitations are described by a **unitary modular tensor category** (UMTC) $\mathcal{M}$.

Gapped walls and condensation completeness

Codimension-one walls, their point-like junctions and local operators naturally form a **monoidal** 2-**category** under fusion.

Whenever condensing point-like defects on a wall produces another gapped wall, we require that the resulting wall and its defect data are included.

This **condensation-complete defect theory** is described by a **fusion** 2-**category** $\mathcal{B}$.

$$\boxed{\Omega\mathcal{B} = \mathrm{End}_{\mathcal{B}}(\mathbf{1}) \cong \mathcal{M}.}$$

Anyons are point-like defects on the trivial wall.

Condensation completion

Gaiotto and Johnson-Freyd developed condensation theory in higher categories.

1. Regard $\mathcal{M}$ as a defect 2-category with a single wall: the **trivial wall**.
2. Adjoin all gapped walls obtained by **condensing anyons** on it, together with their junctions and local operators.
3. The result is the **condensation completion** $\Sigma\mathcal{M}$.

Reconstruction in $(2+1)$D (Johnson-Freyd)

For an anomaly-free bosonic $(2+1)$D topological order,

$$\boxed{\Sigma\mathcal{M} = \Sigma\Omega\mathcal{B} \simeq \mathcal{B}.}$$

D. Gaiotto and T. Johnson-Freyd, *Condensations in higher categories.*
T. Johnson-Freyd, *On the Classification of Topological Orders* (2022), Thm. 4.

A general reconstruction criterion

Let $\mathcal{B}$ be a fusion n-category of bulk defects with trivial E_1-center in $(n+1)$ spacetime dimensions. For $1 \leq p \leq n$, $\Omega^p\mathcal{B}$ retains the defects of codimension $\geq p+1$ and their higher monoidal structure.

Theo Johnson-Freyd's reconstruction criterion

$$\underbrace{\Sigma^p\Omega^p\mathcal{B} \longrightarrow \mathcal{B}}_{\text{canonical monoidal functor}} \quad \text{is an equivalence} \quad \Longleftrightarrow \quad \Omega^{n-p+1}\mathcal{B} \simeq (p-1)\mathbf{Vec}.$$

The right side means that topological defects in spacetime dimensions $< p$ are trivial.

The previous slide is the case $n = 2$, $p = 1$. $\qquad 0\mathbf{Vec} = \mathbb{C}$.

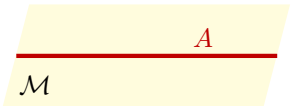
T. Johnson-Freyd, *On the Classification of Topological Orders* (2022), Thms. 4–5 and Rem. IV.4.

The Algebra-Bimodule model of condensation completion

For a **fusion higher category** $\mathcal{C}$, a model of $\Sigma\mathcal{C}$ consists of **separable algebras, bimodules, bimodule maps**, and possible **higher morphisms** in $\mathcal{C}$.

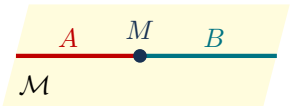
In $(2+1)$D: $\mathcal{B} \simeq \Sigma\mathcal{M}$ for a UMTC $\mathcal{M}$.

Wall



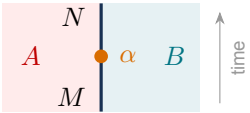
separable algebra A

Junction



B-A bimodule ${}_B M_A$

Local operator



bimodule map $\alpha : M \to N$

Wall fusion $\longleftrightarrow$ tensor product of algebras in $\mathcal{M}$:

$$A \,\square\, B = (A \otimes B,\, m_{A\square B}), \qquad \text{up to Morita equivalence.}$$

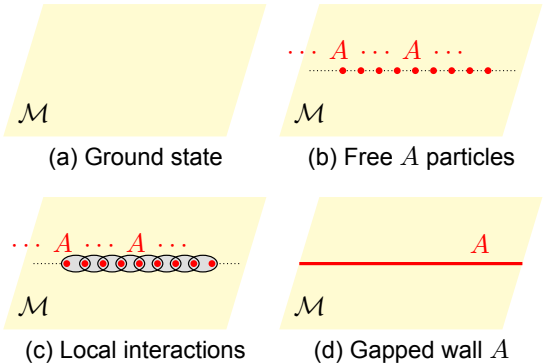
Computing these algebraic data for a given UMTC $\mathcal{M}$ gives all codimension-one defects, as well as their fusions.

G. Yue, L. Wang and T. Lan, *Phys. Rev. B* 112, 104106 (2025).

D. Gaiotto and T. Johnson-Freyd, *Condensations in higher categories*.

From algebra data to a gapped wall

A separable algebra $A \in \mathcal{M}$ specifies the particle sectors and their **local interactions** along the prospective wall.



(a) Ground state (b) Free A particles

(c) Local interactions (d) Gapped wall A

1. Make the A-particle sectors freely available along a line.
2. Couple neighboring sectors using the multiplication m_A and a separability splitting Δ_A.

$$m_A \Delta_A = \mathrm{id}_A, \qquad P_A = \Delta_A m_A.$$

P_A is the local condensation projector.

These interactions produce the gapped wall labelled by A.

G. Yue, L. Wang and T. Lan, *Phys. Rev. B* **112**, 104106 (2025), Fig. 2.

Toric code: the e–m exchange wall

The multiplication m_A and a separability splitting Δ_A encode local wall interactions through the **projector** $\Delta_A m_A$.

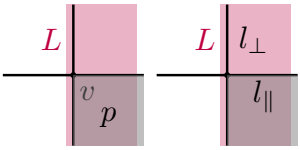
Toric code: $A = \mathbf{1} \oplus f$, $f = e \otimes m$, $f \otimes f = \mathbf{1}$.

1. On a framed line L, replace $-A_v - B_p$ by $-A_v B_p$: the vacuum and f sectors become degenerate.
2. Realize the condensation projector by

$$\Delta_A m_A \;\rightsquigarrow\; P_j = \frac{1+\tau_j}{2}, \qquad \tau_j = \sigma^z_{l_{\perp,j}} \sigma^x_{l_{\parallel,j}}.$$

3. At strong coupling, $\tau_j = 1$ gives the **Kitaev–Kong zigzag wall**.

$$H_{\mathbf{1}\oplus f} = -\sum_{v \notin L} A_v - \sum_{p \notin L} B_p - \sum_{(v,p)\in L} A_v B_p - \lambda \sum_j \tau_j.$$



Framing and the paired edges.

$e \longleftrightarrow m$

$f \longmapsto f$

$A_v = \prod_{l \ni v} \sigma^x_l$, $B_p = \prod_{l \in \partial p} \sigma^z_l$; j labels the paired edges along L.

G. Yue, L. Wang and T. Lan, *Phys. Rev. B* **112**, 104106 (2025).

A. Kitaev and L. Kong, *Models for gapped boundaries and domain walls* (2012).

L. Kong and Z.-H. Zhang, *An invitation to topological orders and category theory* (2022), Sec. 4.3.2.

Finite generalized symmetries

We now consider how to represent symmetries in a condensation-complete defect theory.

In this talk, we consider only finite generalized symmetries.

Symmetry category

In $(n+1)$ spacetime dimensions, such a symmetry is described by a **fusion n-category $\mathcal{T}$**: the category of abstract labels of symmetry operators/defects and their fusion.

SET orders as representations

A symmetry-enriched topological phase

Represent the symmetry in the chosen bare defect theory:

$$\phi : \mathcal{T} \longrightarrow \mathcal{B} \qquad \text{monoidal } n\text{-functor.}$$

- ϕ sends abstract symmetry defects and junctions to physical ones, preserving their fusion and composition.
- Different assignments into the same $\mathcal{B}$ can give different symmetry enrichments.
- The relative center $Z(\phi)$ describes the **charge category**. Physically, it records codimension-one and higher-codimension defects that are transparent to the symmetry defects.

L. Kong and H. Zheng, *Categories of quantum liquids II*, Commun. Math. Phys. 405 (2024) 203.

L. Kong and H. Zheng, *The center functor is fully faithful*, Adv. Math. 339 (2018) 749–779.

Category of SET orders

Ordinary group representations

$$F \in \mathrm{Rep}G = \mathrm{Fun}(BG, \mathbf{Vec}),$$

$$F(\cdot) = V,$$

$$\rho : G \longrightarrow \mathrm{End}_{\mathbf{Vec}}(V).$$

SET representations

$$F \in \mathrm{Fun}(\Sigma\mathcal{T}, \mathcal{X}),$$

$$F(\bullet) = x,$$

$$\phi : \mathcal{T} \longrightarrow \mathcal{B} = \mathcal{X}(x, x).$$

A condensation-complete defect theory (the *bare theory*) is an endomorphism category in a higher vector space $\mathcal{X}$.

$F : \Sigma\mathcal{T} \to \mathcal{X}$ is a higher linear functor, with $\phi = \Omega F$. The higher vector space $\mathcal{X}$ labels the anomaly of the bare theory.

The category of $\mathcal{T}$-SET orders with anomaly $\mathcal{X}$

$$\mathrm{Fun}(\Sigma\mathcal{T}, \mathcal{X})$$

L. Kong and H. Zheng, *Categories of quantum liquids I*, JHEP 08 (2022) 070.

T. Lan, G. Yue and L. Wang, *Category of SET orders*, JHEP 11 (2024) 111.

Generalized gauging

Gauging is, physically, the (partial) condensation of symmetry defects.

Choose the defects to condense, encoded by a nonzero indecomposable separable algebra $A \in \mathcal{T}$. The symmetry assignment realizes it as

$$A \overset{\phi}{\longmapsto} \phi(A) \in \mathcal{B}.$$

The gauged theory

Gauging gives a new symmetry assignment:

$$\boxed{\phi_A : {}_A\mathcal{T}_A \longrightarrow {}_{\phi(A)}\mathcal{B}_{\phi(A)}.}$$

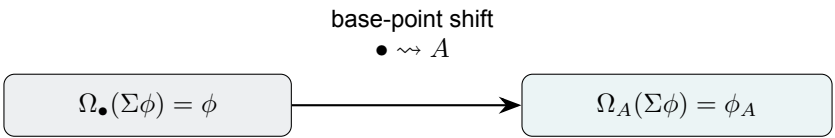
The symmetry is **completely gauged** if the one-sided module category satisfies

$${}_A\mathcal{T} \simeq n\mathbf{Vec} \qquad \text{as a } \mathcal{T}\text{-module.}$$

Gauging as a base-point shift

Condensation completion extends the symmetry assignment to

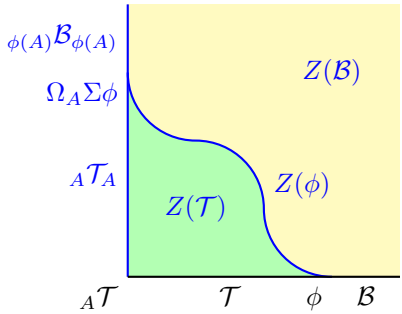
$$\Sigma\phi : \Sigma\mathcal{T} \longrightarrow \Sigma\mathcal{B}, \qquad A \longmapsto \phi(A).$$



- The charge category is preserved: $Z(\phi_A) \simeq Z(\phi)$.
- Gauging is reversible.

Gauging from a sandwich diagram

The gauged symmetry assignment can be encoded by a sandwich diagram.



Close the sandwich formed by

- the dual symmetry ${}_A\mathcal{T}_A$;
- the center (SymTFT) $Z(\mathcal{T})$;
- the charge category $Z(\phi)$.

This gives the gauged theory, equipped with its symmetry assignment ϕ_A.

$$\phi(A)\mathcal{B}_{\phi(A)} \simeq {}_A\mathcal{T}_A \boxtimes_{\overline{Z(\mathcal{T})}} Z(\phi).$$

T. Lan, G. Yue and L. Wang, *Category of SET orders*, JHEP 11 (2024) 111, Fig. 3 (right).

L. Kong and H. Zheng, *The center functor is fully faithful*, Adv. Math. 339 (2018) 749–779.

Gauging $(1+1)$D $\mathrm{Rep}G$ SPT

Consider the SPT with the forgetful symmetry assignment $\phi_0 : \mathrm{Rep}(G) \to \mathbf{Vec}$. Since it is an SPT, its bare defect theory is $\mathbf{Vec}$.

Morita classes of indecomposable separable algebras in $\mathrm{Rep}(G)$ are classified by pairs $H \leq G$, $[\omega] \in H^2(H, U(1))$, up to G-conjugacy. A representative of each class is

$$A(H,\omega) = \mathrm{Ind}_H^G(W \otimes W^*), \qquad d = \dim W, \quad n = [G:H].$$

Here $\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ is induction, and W is an irreducible ω-projective representation of H.

The gauged theory

$$\phi_0\big(A(H,\omega)\big) \cong M_d(\mathbb{C})^{\oplus n},$$

$${}_{\phi_0(A)}\mathbf{Vec} \simeq \mathbf{Vec}^{\oplus n}, \qquad N_{\mathrm{vac}} = n = \frac{|G|}{|H|} = \dim Z\big(\phi_0(A)\big),$$

$${}_{\phi_0(A)}\mathbf{Vec}_{\phi_0(A)} \simeq_{\otimes} \mathrm{Mat}_n(\mathbf{Vec}).$$

Gauging $(1+1)$D $\mathrm{Rep}D_8$ SPT

Gauging the forgetful SPT $\phi_0 : \mathrm{Rep}(D_8) \to \mathbf{Vec}$:

Algebra A	Underlying object	$\phi_0(A)$	${}_A\mathrm{Rep}(D_8)_A$	${}_{\phi_0(A)}\mathrm{Vec}_{\phi_0(A)}$	N_{vac}
$\mathrm{Fun}(D_8)$	$\bigoplus_\chi \chi \oplus D^{\oplus 2}$	$\mathbb{C}^8$	Vec_{D_8}	$\mathrm{Mat}_8(\mathrm{Vec})$	8
$\mathrm{Fun}(D_8/\langle a^2\rangle)$	$\bigoplus_\chi \chi$	$\mathbb{C}^4$	$\mathrm{Vec}_{\mathbb{Z}_2^3}^{\Omega}$	$\mathrm{Mat}_4(\mathrm{Vec})$	4
$\mathrm{Fun}(D_8/\langle x\rangle)$	$\mathbf{1} \oplus \eta_a \oplus D$	$\mathbb{C}^4$	$\mathrm{Rep}(D_8)$	$\mathrm{Mat}_4(\mathrm{Vec})$	4
$\mathrm{Fun}(D_8/\langle ax\rangle)$	$\mathbf{1} \oplus \eta \oplus D$	$\mathbb{C}^4$	$\mathrm{Rep}(D_8)$	$\mathrm{Mat}_4(\mathrm{Vec})$	4
$\mathrm{Fun}(D_8/\langle a\rangle)$	$\mathbf{1} \oplus \eta_x$	$\mathbb{C}^2$	Vec_{D_8}	$\mathrm{Mat}_2(\mathrm{Vec})$	2
$\mathrm{Fun}(D_8/K_x)$	$\mathbf{1} \oplus \eta_a$	$\mathbb{C}^2$	Vec_{D_8}	$\mathrm{Mat}_2(\mathrm{Vec})$	2
$A(K_x, -1)$	$\bigoplus_\chi \chi \oplus D^{\oplus 2}$	$M_2(\mathbb{C})^{\oplus 2}$	Vec_{D_8}	$\mathrm{Mat}_2(\mathrm{Vec})$	2
$\mathrm{Fun}(D_8/K_{ax})$	$\mathbf{1} \oplus \eta$	$\mathbb{C}^2$	Vec_{D_8}	$\mathrm{Mat}_2(\mathrm{Vec})$	2
$A(K_{ax}, -1)$	$\bigoplus_\chi \chi \oplus D^{\oplus 2}$	$M_2(\mathbb{C})^{\oplus 2}$	Vec_{D_8}	$\mathrm{Mat}_2(\mathrm{Vec})$	2
$\mathbf{1}$	$\mathbf{1}$	$\mathbb{C}$	$\mathrm{Rep}(D_8)$	Vec	1
$\mathrm{End}_{\mathbb{C}}(D)$	$\bigoplus_\chi \chi$	$M_2(\mathbb{C})$	$\mathrm{Rep}(D_8)$	Vec	1
$A(D_8, -)$	$\mathbf{1} \oplus \eta_x \oplus D$	$M_2(\mathbb{C})$	$\mathrm{Rep}(D_8)$	Vec	1

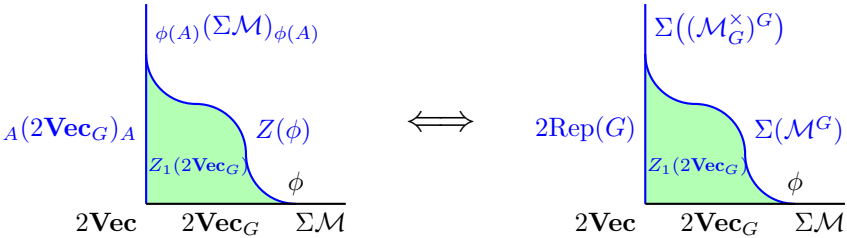
$D_8 = \langle a, x \mid a^4 = x^2 = e,\ xax = a^{-1}\rangle$, $K_x = \langle a^2, x\rangle$, $K_{ax} = \langle a^2, ax\rangle$; χ runs over the four one-dimensional representations.

- If $\phi_0(A)$ is a matrix algebra, the SPT is self-dual under gauging; this agrees with the lattice example of SSY.
- Green: the triality orbit of the inverse-KW gauging algebra in Cao–Yamazaki–Li.

S. Seifnashri, S.-H. Shao and X. Yang, *Gauging non-invertible symmetries on the lattice* (2025).
W. Cao, M. Yamazaki and L. Li, *Duality viewpoint of noninvertible symmetry protected topological phases* (2026).
Diatlyk–Luo–Wang–Weller (DLWW23), *Gauging Non-Invertible Symmetries*, arXiv:2311.17044, Secs. 3.4–3.5, Table 2.
G. Yue, PhD thesis, The Chinese University of Hong Kong (in preparation).

Complete gauging: $(2+1)$D bosonic G-SET

$$\phi : 2\mathbf{Vec}_G \longrightarrow \Sigma\mathcal{M}, \qquad A = \mathbf{Vec}_G \in 2\mathbf{Vec}_G.$$



- $\mathcal{M}^G$ is the category of charged anyons; $\Sigma(\mathcal{M}^G)$ is its condensation completion.
- ϕ determines the anomaly-free G-action and its canonical G-crossed extension $\mathcal{M}_G^\times$ of $\mathcal{M}$.
- $(\mathcal{M}_G^\times)^G$ is a minimal modular extension of $\mathcal{M}^G$.

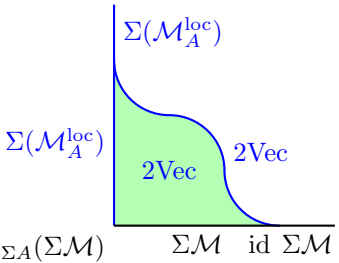
Our gauging procedure is compatible with gauging in the literature.

T. Lan, G. Yue and L. Wang, *Category of SET orders*, JHEP 11 (2024) 111, Fig. 8.
Lan–Kong–Wen, *Modular extensions of unitary braided fusion categories and 2+1D topological/SPT orders with symmetries* (2017).
Etingof–Nikshych–Ostrik, *Fusion categories and homotopy theory*, Quantum Topol. 1 (2010).
Drinfeld–Gelaki–Nikshych–Ostrik, *On braided fusion categories I*, Selecta Math. 16 (2010).
Barkeshli–Bonderson–Cheng–Wang, *Symmetry fractionalization, defects, and gauging of topological phases*, PRB 100 (2019).
G. Yue, PhD thesis, The Chinese University of Hong Kong (in preparation).

Gauging canonical 1-symmetry in $(2+1)$D $=$ Anyon Condensation

- An n-symmetry is generated by condensation from symmetry operators of codimension $n+1$.

$$\mathrm{id} : \Sigma\mathcal{M} \longrightarrow \Sigma\mathcal{M}, \qquad A \in \mathcal{M} \text{ a condensable algebra.}$$



$$\boxed{{}_{\Sigma A}(\Sigma\mathcal{M})_{\Sigma A} \cong \Sigma(\mathcal{M}_A^{\mathrm{loc}})}$$

- $\mathcal{M}_A^{\mathrm{loc}}$ is the category of local A-modules in $\mathcal{M}$.
- Physically, it is the remaining anyon theory after condensing A.

L. Kong, *Anyon condensation and tensor categories*, Nucl. Phys. B 886 (2014) 436–482.

L. Kong, Z.-H. Zhang, J. Zhao and H. Zheng, *Higher condensation theory*, arXiv:2403.07813v3, Theorem 5.1.5.

Thanks for listening!