# Mathematical Research with AI: $k$-Generalized Markov Equations

This talk differs slightly from what I outlined in the abstract.

**Yasuaki Gyoda**
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# Outline

1. **Earlier work and my goal**
2. Progress
3. How I used AI

# $k$-generalized Markov ($k$-GM) equations

**$k$-GM equation** (Gyoda–Matsushita, 2023)

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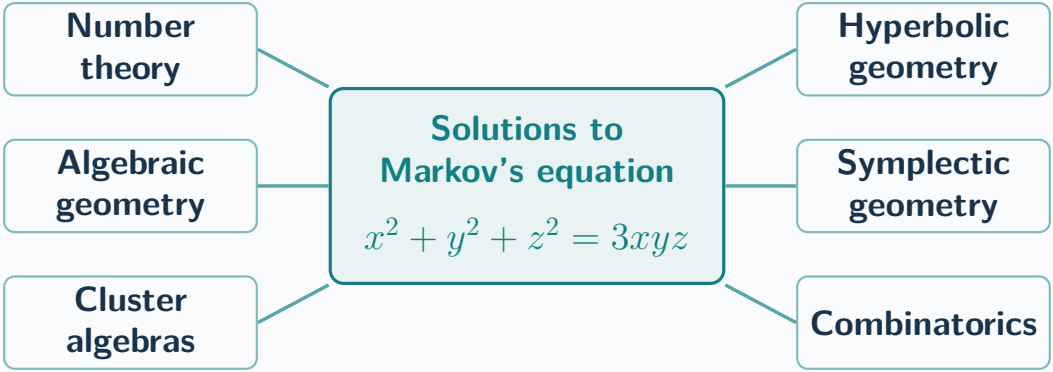
Each positive integer solution is called a **$k$-GM triple**.

Its entries are called **$k$-GM numbers**.

| Value of $k$ | Positive integer solutions $(x, y, z)$ | | | |
|---|---|---|---|---|
| 0 | $(1, 1, 1)$ | $(1, 1, 2)$ | $(1, 2, 5)$ | ... |
| 1 | $(1, 1, 1)$ | $(1, 1, 3)$ | $(1, 3, 13)$ | ... |
| 2 | $(1, 1, 1)$ | $(1, 1, 4)$ | $(1, 4, 25)$ | ... |

# Applications



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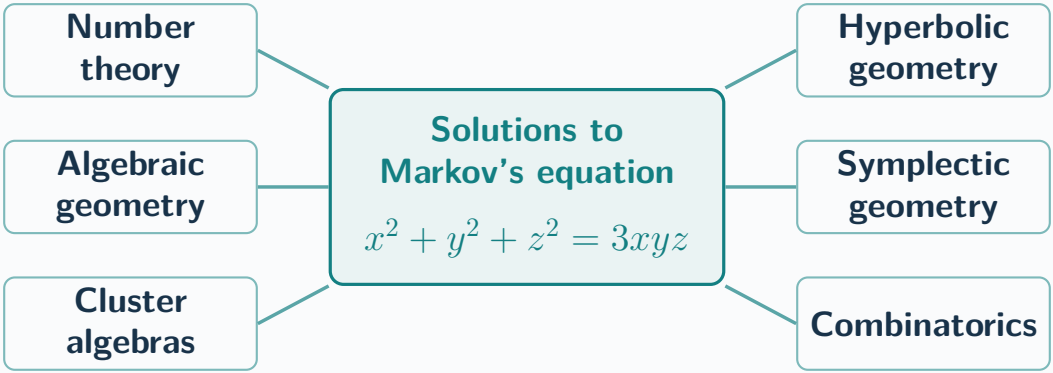


**We aimed to extend these links to $k$-GM equations for $k > 0$.**

# The Markov conjecture and its generalization

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For every Markov (0-GM) number $n$, there is exactly one Markov triple $(x, y, z)$ with $0 < x \leq y \leq z = n$.

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**$k$-GM uniqueness conjecture** (Gyoda–Maruyama, 2026)

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For $k = 0$ and $n = 5$, $(1, 2, 5)$ solves $x^2 + y^2 + z^2 = 3xyz$.

For $k = 1$ and $n = 13$, $(1, 3, 13)$ solves $x^2 + y^2 + z^2 + xy + yz + zx = 6xyz$.

Each is the only solution in integers $0 < x \leq y \leq z$ with largest entry $n$.

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# Previous work: The Gaster–Loustau identity

**Theorem** (Gaster–Loustau, 2021)

Let $\mathcal{M}_0$ be the set of distinct Markov numbers. For each $n$ in this set, its Lagrange number and multiplicity are

$$\mathcal{L}_0(n) = \sqrt{9 - \frac{4}{n^2}}, \qquad \mu_0(n) = \begin{cases} 3 & n \in \{1, 2\}, \\ 6 & \text{otherwise.} \end{cases}$$

The Markov conjecture is true **if and only if**

$$\sum_{n \in \mathcal{M}_0} \mu_0(n)\big(3 - \mathcal{L}_0(n)\big) = 3.$$

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Lagrange numbers measure how well rationals approximate irrationals. The theorem suggests a link between the Markov conjecture and Diophantine approximation, a branch of number theory.

# The generalized Gaster–Loustau identity

**Theorem** (Gyoda, 2026)

Fix an integer $k \geq 0$. Let $\mathcal{M}_k$ be the set of distinct $k$-GM numbers. For each $n$ in this set, its Lagrange number and multiplicity are

$$\mathcal{L}_k(n) = \frac{\sqrt{\big(3(1+k)n - k\big)^2 - 4}}{n}, \qquad \mu_k(n) = \begin{cases} 3 & n \in \{1, k+2\}, \\ 6 & \text{otherwise.} \end{cases}$$

The $k$-GM uniqueness conjecture is true **if and only if**

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When $k = 0$, we get the original Gaster–Loustau theorem.

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**The tool I used**

**ChatGPT** — GPT-6 Astra and GPT-6.1 Sol

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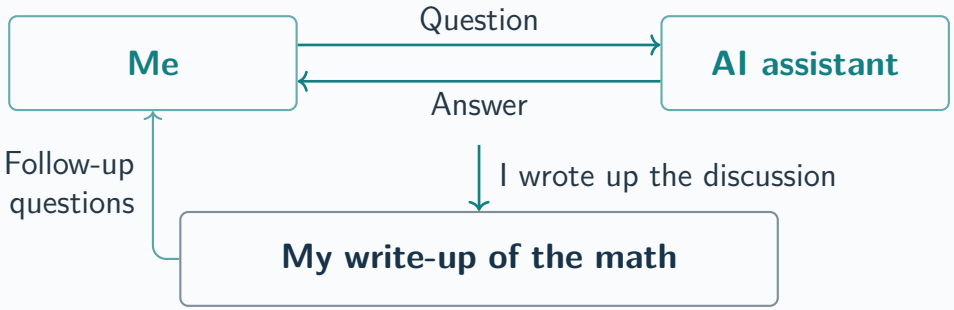
**The tool I used**

**ChatGPT** — GPT-6 Astra and GPT-6.1 Sol

I asked one question at a time. Each answer led to the next question.



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**Step 4. Wrote and proofread a paper (10 days)** I wrote the paper and checked the draft for mistakes and unclear wording.

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- I wanted to find a theorem I really cared about.
- If AI found a theorem in a field I knew little about, I might not see why it mattered or how to use it. Then I'd have little reason to publish a paper on it myself.
- So I asked AI to look for theorems close to my own research interests. I limited this project to extending a known theorem about classical Markov numbers to $k$-GM numbers.

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- I added useful facts and proofs to my draft.
- I kept doing this to learn the existing theory and look for new results.

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- I asked ChatGPT to explain any step I didn't understand. I kept asking until I could follow the whole proof.

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- I tried the second way. Revising the AI draft took a lot of work. I thought it would have been better to work through the ideas and write the first draft myself.

# Conclusion

- Given a theorem to prove, AI does at least as well as humans.
- AI seems about as good as humans at generating innovative ideas.
- Humans are clearly better than AI at writing research papers.

These are my conclusions as of November 2026. New evidence could overturn them in the next few years. For now, writing a research paper still requires a human expert in the field.

**Thank you for your attention.**