Conditional Independence of a Beam Splitter Output with Quantum Memory

A conditional quantum Kac–Bernstein theorem

Hesam Hosseini (Sam)

Joint work with Jinpei Zhao and Masahito Hayashi

Department of Information Engineering, CUHK

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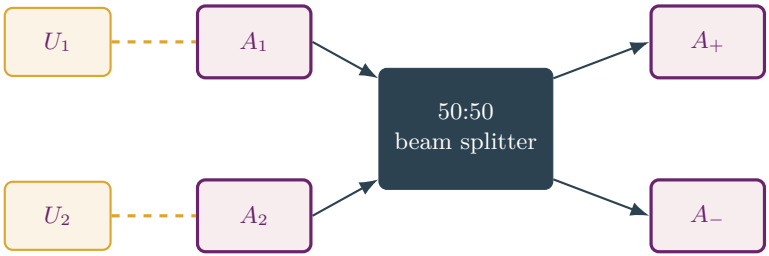
The question: independence after a beam splitter



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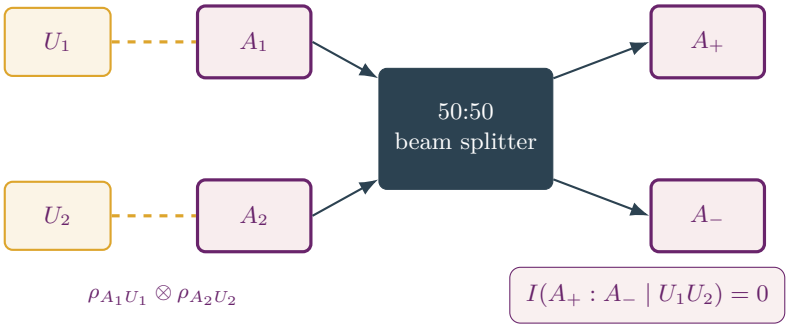

What does zero conditional mutual information force?

The two input–memory pairs are **independent, not necessarily identical**. Each pair may be correlated or entangled. We classify **all** equality inputs.

Motivation: Conditional Gaussian optimality

Let $f(X \mid U)$ be a functional of a physical system X with side information U. Assume three properties:

1. **Strict conditional subadditivity:** $f(X_1X_2 \mid U) \leq f(X_1 \mid U) + f(X_2 \mid U)$, with equality iff $I(X_1 : X_2 \mid U) = 0$.
2. **Rotation invariance:** $f(\mathcal{R}X \mid U) = f(X \mid U)$ for an admissible physical rotation $\mathcal{R}$.
3. **Compatibility with independent systems:** independent side information leaves f unchanged, and independent input–memory pairs are additive:

$$f(X \mid UU') = f(X \mid U) \quad (U' \perp XU),$$
$$f(X_1X_2 \mid U_1U_2) = f(X_1 \mid U_1) + f(X_2 \mid U_2).$$



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Examples and motivation

1. **Conditional von Neumann entropy:** $f(A \mid U) = S(A \mid U)$.
2. **Conditional entropy power inequality (EPI)** and its Lieb-type formulation

The conditional doubling–rotation argument

Goal: Maximize $f(X \mid U)$ subject to $\mathrm{Cov}(X) \preceq K$. Let (X^*, U^*) attain the maximum v. Take two independent copies (X_1, U_1), (X_2, U_2) and set

$$U = U_1 U_2, \qquad X_\pm = \frac{X_1 \pm X_2}{\sqrt{2}}.$$



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By independent-pair additivity, rotation invariance and subadditivity,

$$\begin{aligned} 2v &= f(X_1 \mid U_1) + f(X_2 \mid U_2) \\ &= f(X_1 X_2 \mid U) \\ &= f(X_+ X_- \mid U) \\ &\leq f(X_+ \mid U) + f(X_- \mid U) \leq 2v. \end{aligned}$$

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Equality throughout forces

$$I(X_+ : X_- \mid U_1 U_2) = 0.$$

Characterization

The conditional Kac–Bernstein theorem identifies the **equality-state structure** imposed on such maximizers.



Classical Kac–Bernstein: the benchmark

Let $X_1, X_2 \in \mathbb{R}^d$ be independent, and define

$$S = \frac{X_1 + X_2}{\sqrt{2}}, \qquad D = \frac{X_1 - X_2}{\sqrt{2}}.$$

Independence of S, D gives the Bernstein equation for $\varphi_i(s) = \mathbb{E}[e^{\mathrm{i}s^\top X_i}]$:

$$\boxed{\varphi_1(s+t)\varphi_2(s-t) = \varphi_1(s)\varphi_2(s)\varphi_1(t)\varphi_2(-t).}$$

Gaussianity. X is Gaussian when its characteristic function is

$$\boxed{\varphi_X(s) = e^{\mathrm{i}m^\top s - \frac{1}{2}s^\top \Sigma s},}$$

where m and Σ are its mean and covariance.



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Conditional classical extension. For independent pairs (X_1, U_1) and (X_2, U_2),

$$I(S : D \mid U_1 U_2) = 0 \quad \Longrightarrow \quad \boxed{X_i \mid U_i = u_i \sim \mathcal{N}(b_i(u_i), \Sigma).}$$

The covariance Σ is common and independent of u_i.



Quantum Kac–Bernstein: the challenge

For n-mode bosonic inputs ρ_{A_i}, let $d = 2n$, $R_i = (Q_{i,1}, P_{i,1}, \ldots, Q_{i,n}, P_{i,n})^{\mathsf{T}}$. The **quantum characteristic function** is

$$\chi_{\rho_i}(s) = \mathrm{Tr}[\rho_{A_i} W_i(s)], \qquad W_i(s) = e^{\mathrm{i} s^{\mathsf{T}} R_i}, \quad s \in \mathbb{R}^d.$$

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Here b is the mean, Γ the covariance, and Ω the canonical symplectic matrix.



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Independent beam-splitter outputs imply **Gaussian input states with a common covariance**, up to displacements.

Now introduce quantum memories. Each A_i may be correlated or entangled with an independent memory U_i. The condition becomes

$$\boxed{I(A_+ : A_- \mid U_1 U_2) = 0.}$$

Central question

Does Gaussianity survive conditioning on **noncommuting quantum memories**?



Quantum conditional independence

Quantum information measures

$$S(\rho) = -\operatorname{Tr}(\rho \log \rho) \quad S(A \mid U) = S(AU) - S(U), \quad I(A : U) = S(A) + S(U) - S(AU).$$

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For a tripartite state ζ_{BMC}, the **conditional mutual information (CMI)** is

$$\boxed{I(B : C \mid M)_\zeta = S(BM) + S(CM) - S(M) - S(BCM) \geq 0.}$$



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Zero CMI: $I(B : C \mid M) = 0$ means $B - M - C$ is a **quantum Markov chain**.

Quantum Markov decomposition

$$\mathcal{H}_M \simeq \bigoplus_a \mathcal{H}_{L_a} \otimes \mathcal{H}_{R_a},$$

$$\zeta_{BMC} \simeq \bigoplus_a r_a \, \eta^a_{BL_a} \otimes \theta^a_{R_a C}.$$

Here r_a are probabilities, and η^a, θ^a are states; the decomposition is on $\operatorname{supp} \zeta_M$.

Infinite-dimensional: Jenčová–Petz (2006), with marginal-support extension in our Lemma 2.1;
Finite-dimensional: Hayden–Jozsa–Petz–Winter (2004).



What structure should we expect?

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Classical intuition. The classical conditional theorem gives $X \mid U = u \sim \mathcal{N}(b(u), \Sigma)$. The joint law is a Gaussian mixture:

$$\boxed{P_{XU}(dx, du) = P_U(du)\,\mathcal{N}(b(u), \Sigma)(dx).}$$



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Equivalently,

$$\boxed{X = b(U) + Z, \qquad Z \sim \mathcal{N}(0, \Sigma), \quad Z \perp U.}$$

Its conditional characteristic function factorizes:

$$\mathbb{E}[e^{is^\top X} \mid U] = \underbrace{e^{-\frac{1}{2}s^\top \Sigma s}}_{\text{Gaussian core}} \underbrace{e^{is^\top b(U)}}_{\text{memory-dependent displacement}}.$$

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Guiding intuition for the quantum theorem

Can we identify a **common Gaussian core** and **memory-dependent displacements** directly in $\rho_{A_i U_i}$?

Precise setting and beam-splitter action

Independent input pairs: $\rho_1 \otimes \rho_2$, with $\rho_i = \rho_{A_i U_i}$. Each A_i has $n < \infty$ modes ($d = 2n$); the U_i are arbitrary separable Hilbert-space memories. The output state is $\omega = (B \otimes I)(\rho_1 \otimes \rho_2)(B^* \otimes I)$.

Assumptions: finite physical energy and memory entropy:

$$\mathrm{Tr}(\rho_{A_i} H_{A_i}) < \infty, \qquad S(\mu_i) < \infty, \quad \mu_i = \mathrm{Tr}_{A_i}\, \rho_i.$$

Here $\rho_{A_i} = \mathrm{Tr}_{U_i}\, \rho_i$ and H_{A_i} is the photon-number Hamiltonian.



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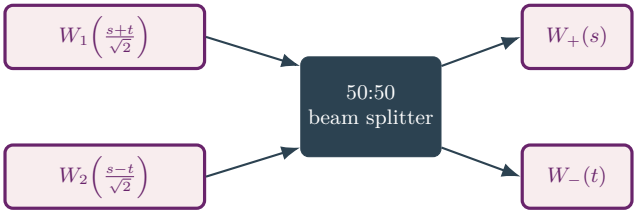
Beam-splitter action on Weyl operators



Heisenberg-picture transformation:

$$B^*\big(W_+(s) \otimes W_-(t)\big)B = W_1\left(\frac{s+t}{\sqrt{2}}\right) \otimes W_2\left(\frac{s-t}{\sqrt{2}}\right)$$

Operator-valued characteristic functions

For ρ_{AU}, define the **partial characteristic operator** (on the memory Hilbert space):

$$\Phi(s) = \mathrm{Tr}_A[\rho_{AU}(W_A(s) \otimes I_U)], \qquad \Phi(0) = \mu := \mathrm{Tr}_A\, \rho_{AU}.$$

On $\mathrm{supp}\, \mu$, define the **normalized family** $K(s)$ by

$$\boxed{\Phi(s) = \mu^{1/2} K(s) \mu^{1/2}}, \qquad K(0) = I, \quad \|K(s)\| \leq 1.$$

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Modular evolution of a memory operator X is

$$\alpha_u(X) = \mu^{\mathrm{i}u} X \mu^{-\mathrm{i}u}, \qquad K^{(u)}(s) = \alpha_u(K(s)), \quad u \in \mathbb{R}.$$

For $\mu = \sum_j p_j |j\rangle\langle j|$ (on its support), letting $X = |j\rangle\langle k|$ gives

$$\boxed{\alpha_u(X) = e^{\mathrm{i}u(\log p_j - \log p_k)} X.}$$



Main result: Conditional quantum Kac–Bernstein

Theorem 3.1 (Hosseini–Zhao–Hayashi)

Under the stated assumptions, the following are **equivalent**:

1. **Conditional independence:** $I(A_+ : A_- \mid U_1U_2)_\omega = 0$.



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1. **Conditional independence:** $I(A_+ : A_- \mid U_1 U_2)_\omega = 0$.
2. **Operator characterization:** K_1, K_2 satisfy (B) and (M), for $s, t \in \mathbb{R}^d$, $u, v \in \mathbb{R}$:

$$\text{(B)} \quad K_1(s+t) \otimes K_2(s-t) = K_1(s)K_1(t) \otimes K_2(s)K_2(-t),$$

$$\text{(M)} \quad \left[K_1^{(u)}(s) \otimes K_2^{(u)}(s), K_1^{(v)}(t) \otimes K_2^{(v)}(-t)\right] = 0.$$

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3. **Common Gaussian normal form:** A common bosonic memory factor Q has $\mathcal{H}_Q = \mathcal{F}(\ell^2(\mathcal{I}; \mathbb{C}))$, and

$$\mathcal{H}_{U_i} \simeq \mathcal{H}_Q \otimes \Big(\bigoplus_{j \in \mathcal{J}_i} \mathcal{H}_{K_j^{(i)}}\Big),$$

$$\rho_i \simeq \bigoplus_{j \in \mathcal{J}_i} p_j^{(i)} \, \gamma_{AQ}^{b_j^{(i)}} \otimes \nu_j^{(i)}.$$

γ_{AQ} is the **common centered Gaussian core**; γ_{AQ}^b displaces only A. The $p_j^{(i)}$ are sector weights, and $\nu_j^{(i)}$ residual states.

Paper: Theorem 3.1, Eqs. (3.1)–(3.3).



Proof I: From Markov structure to operator equations

Step 1: Quantum Markov factorization Let $\mu = \mu_1 \otimes \mu_2$. Normalize the joint Weyl coefficient:

$$\mathrm{Tr}_{A_+A_-}[\omega(W_+(s) \otimes W_-(t) \otimes I)] = \mu^{1/2} F(s,t) \mu^{1/2}.$$

Zero CMI and the Markov decomposition imply

$$\boxed{F(s,t) = F(s,0)F(0,t),}$$

with modular commutation of $F(s,0)$ and $F(0,t)$.



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Resulting operator equations

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Proof II: Gaussian rigidity and Weyl structure

Step 3: Tensor proportionality For nonzero simple tensors,

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Step 4: Gaussian rigidity and Weyl structure Applying tensor proportionality to (B), operator-norm constraints, and modular commutation (M) yields

$$\boxed{\begin{aligned} K_i(s) &= e^{-\frac{1}{4} s^{\mathsf{T}} G s} V_i(s), \\ V_i(s) V_i(t) &= e^{-\frac{\mathrm{i}}{2} s^{\mathsf{T}} \Theta t} V_i(s+t), \\ V_i^{(u)}(s) V_i^{(v)}(t) &= e^{-\mathrm{i} s^{\mathsf{T}} C_{uv} t} V_i^{(v)}(t) V_i^{(u)}(s). \end{aligned}}$$

Here $V_i(s)$ are unitary, $V_i^{(u)}(s) := \alpha_u^{(i)}(V_i(s))$; $G \succeq 0$, $\Theta = -\Theta^{\mathsf{T}}$, and the real matrices G, Θ, C_{uv} are **common to both inputs**.

Paper: Proposition 4.3.



Stone–von Neumann: from Weyl relations to CCR

1. From position–momentum to Weyl operators On $L^2(\mathbb{R})$, the position and momentum operators are

$$(Q\psi)(x) = x\psi(x), \qquad (P\psi)(x) = -\mathrm{i}\frac{d}{dx}\psi(x).$$

They satisfy $[Q, P] = \mathrm{i}I$. For $W(s,t) = e^{\mathrm{i}(sQ+tP)}$,

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3. Stone–von Neumann theorem

Every regular irreducible finite-mode Weyl representation is, up to unitary equivalence, the **standard position–momentum representation**.



From Bernstein operators to bosonic modes

The Bernstein equation yields

$$V_i(s)V_i(t) = e^{-\frac{\mathrm{i}}{2}s^{\mathsf{T}}\Theta t}V_i(s+t),$$

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Our strategy

Construct the **modular covariance space** E, separate commuting and canonical directions, and reconstruct the common bosonic memory factor.



Modular covariance space and Weyl representation

By Stone's theorem, $V(te_k) = e^{\mathrm{i}tA_k}$. Set $b_{0,k} = \mathrm{Tr}(\rho_A R_k)$ and $B_{k,u} = \mu^{\mathrm{i}u}(A_k - b_{0,k}I)\mu^{-\mathrm{i}u}$. Let $\mathcal{D}$ be freely spanned by $[k,u]$; for $a = \sum c_{k,u}[k,u]$, set $F_a = \sum c_{k,u}B_{k,u}$. The Hilbert–Schmidt form splits into covariance and commutation:

$$\langle F_a\mu^{1/2}, F_b\mu^{1/2}\rangle_{\mathrm{HS}} = g(a,b) + \frac{\mathrm{i}}{2}\sigma_E(a,b), \qquad E = \overline{\mathcal{D}/\ker g}^{\,g}.$$

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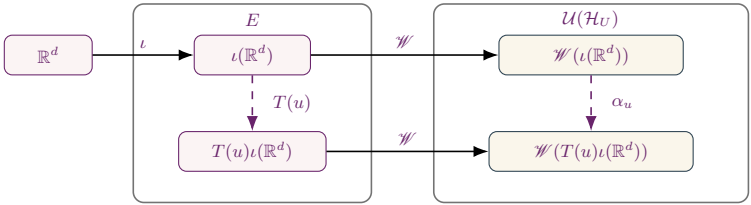

$\iota e_k = [k,0]$ and $T(u)$ shifts modular time.

Weyl representation on E:

$$\boxed{\begin{aligned}&\mathscr{W}(a)\mathscr{W}(b) = e^{-\mathrm{i}\sigma_E(a,b)/2}\mathscr{W}(a+b),\\ &\alpha_u(\mathscr{W}(a)) = \mathscr{W}(T(u)a).\end{aligned}}$$

Fixed and rotating memory directions

Modular evolution is an orthogonal group $T(u)$ on E. The eigenvalues p_j of μ give frequencies $\omega = \log p_j - \log p_k$, and hence

$$\boxed{E = E_0 \oplus \bigoplus_{\omega > 0} E_\omega .}$$



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$$T(u)a = a$$
$$E_0 = \ker \sigma_E$$

These directions commute with **every field**.

Rotating E_ω

$$T(u)|_{E_\omega} = \cos(\omega u) I + \sin(\omega u) J_\omega$$
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The two memories have the same oscillators

The common relation $V_i^{(u)}(s)V_i^{(v)}(t) = e^{-\mathrm{i}s^{\mathsf{T}}C_{uv}t}V_i^{(v)}(t)V_i^{(u)}(s)$ induces an onto isometry

$$\mathcal{S} : E_1^{\mathrm{rot}} \longrightarrow E_2^{\mathrm{rot}}$$

preserving the symplectic form, modular evolution and physical couplings. Thus the rotating frequencies and multiplicities are **identical**.

Paper: Lemma 5.3 and Proposition 5.4.



From canonical pairs to memory factorization

Canonical pairs. On E_ω, choose orthonormal $e_\ell, -J_\omega e_\ell$ and $\kappa_\omega = 2\tanh(\omega/2)$:

$$\mathfrak{q}_\ell = \kappa_\omega^{-1/2} e_\ell, \qquad \mathfrak{p}_\ell = -\kappa_\omega^{-1/2} J_\omega e_\ell.$$

Stone's theorem defines Q_ℓ, P_ℓ by $\mathscr{W}(t\mathfrak{q}_\ell) = e^{\mathrm{i}tQ_\ell}$, $\mathscr{W}(t\mathfrak{p}_\ell) = e^{\mathrm{i}tP_\ell}$, giving

$$\boxed{\sigma_E(\mathfrak{q}_\ell, \mathfrak{p}_m) = \delta_{\ell m}, \quad [Q_\ell, P_m] = \mathrm{i}\delta_{\ell m} I.}$$

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Common bosonic memory. The CCR construction yields

$$\boxed{\begin{aligned} \mathcal{H}_{U_i} &\simeq \mathcal{H}_Q \otimes \mathcal{H}_{K_i}, \qquad \mu_i = \tau_Q \otimes \nu_i, \\ \mathcal{H}_Q = \mathcal{F}(\ell^2(\mathcal{I};\mathbb{C})), \quad \tau_Q &= \bigotimes_{\ell\in\mathcal{I}} \tau(q_\ell), \quad q_\ell = e^{-\omega_\ell}. \end{aligned}}$$

$\tau(q) = (1-q)\sum_{m\geq 0} q^m |m\rangle\langle m|$ is a one-mode thermal state.

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$\tau(q) = (1-q)\sum_{m\geq 0} q^m |m\rangle\langle m|$ is a one-mode thermal state.

Original Weyl operators. The rotating and fixed directions give

$$\boxed{V_i(s) = W_Q(Ls) \otimes Z_i(s),}$$

where $L : \mathbb{R}^d \to \ell^2(\mathcal{I};\mathbb{R}^2)$ is common, $Z_i(s)Z_i(t) = Z_i(s+t)$, and $[Z_i(s), \nu_i] = 0$. Thus W_Q is the **bosonic component** and Z_i the **commuting residual component**.

Paper: Proposition 6.2.



Reconstructing the common Gaussian core

Step 1: Reconstruct the characteristic operator Combining the previous results gives

$$\Phi_i(s) = \bigoplus_j p_j^{(i)} e^{-\frac{1}{4}s^{\mathsf{T}}Gs + \mathrm{i}(b_j^{(i)})^{\mathsf{T}}s} \tau_Q^{1/2} W_Q(Ls) \tau_Q^{1/2} \otimes \nu_j^{(i)}.$$



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Step 2: Identify the Gaussian covariance For each sector j and finite mode subset $F \subset \mathcal{I}$, the joint Weyl characteristic function is Gaussian, with

$$\boxed{\Gamma_F = \begin{pmatrix} G + L^{\mathsf{T}}\mathsf{V}L & L_F^{\mathsf{T}}\mathsf{C}_F \\ \mathsf{C}_F L_F & \mathsf{V}_F \end{pmatrix}.}$$

Here $\mathsf{V} = \bigoplus_\ell v_\ell I_2$ and $\mathsf{C} = \bigoplus_\ell c_\ell I_2$, where $v_\ell = (1+q_\ell)/(1-q_\ell)$, $c_\ell = 2\sqrt{q_\ell}/(1-q_\ell)$.



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The common Gaussian normal form

The covariance is **independent of input i and sector j**. Weyl uniqueness gives

$$\boxed{\rho_i \simeq \bigoplus_{j \in \mathcal{J}_i} p_j^{(i)} \, \gamma_{AQ}^{b_j^{(i)}} \otimes \nu_j^{(i)}.}$$

Common: γ_{AQ}, τ_Q, L. **Variable:** $p_j^{(i)}$, $b_j^{(i)}$, $\nu_j^{(i)}$.

Paper: Section 7.1, Eqs. (7.2)–(7.6).



Selected references and Q&A

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