Wakimoto realization for Lie superalgebras and SUSY analogues

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References

- Genra – I. – Nakatsuka, "Wakimoto realizations for basic classical simple Lie superalgebras", In preparation,
- I., "Free field realizations of $N=1$ SUSY affine Lie superalgebras", preprint, arXiv : 2608.22340.

§Introduction

What is free field realization (or Wakimoto module)?

Answer : to realize the aff. Lie alg. $\hat{\mathfrak{g}}$ by simpler op's (more precisely, free fields).

Finite dimensional.

$\mathfrak{g}$: simple Lie alg., G : alg. grp. corresp. to $\mathfrak{g}$,
$B_\pm$: (opposite) Borel, $N_\pm$: unip. rad of $B_\pm$

$$B_- \backslash G \curvearrowleft G$$

$$\rightsquigarrow \rho : \mathfrak{g} \longrightarrow \mathrm{Vect}(U),$$

where

- $B_-\backslash G \overset{\text{open}}{\supset} U = B_-\backslash B_- N_+ \simeq \mathbb{A}^{\Delta_+}$: big cell.
- $\mathrm{Vect}(U) = \{\text{deriv's on } U\}$.

Exm. ($\mathfrak{g} = \mathfrak{sl}_2$)

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \longmapsto a,$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \longmapsto -2a^* a,$$

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \longmapsto -a^{*2} a,$$

where

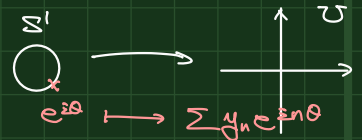
$a^* = y$: coordinate on U,
$a = \partial_y$: derivation of y.

Loop analogue

For simplicity, we only explain the case of $\mathfrak{g} = \mathfrak{sl}_2$.

$U = \mathbb{A}^1_{y_\bullet}$,

$\mathcal{L}U = \left\{ \sum_{n \gg -\infty} y_n z^n \right\} = \bigcup_{N \in \mathbb{Z}} \{ (y_n) \mid n \geq N \}$.



Then,

$$\rho : \mathfrak{g} \longrightarrow \mathrm{Vect}(U)$$

$\rightsquigarrow \mathcal{L}\rho : \mathfrak{g}((t)) \longrightarrow \mathrm{Vect}(\mathcal{L}U)$: hom. of Lie alg's.

This hom. is explicitely written as

$$e(z) \longmapsto a(z),$$
$$h(z) \longmapsto -2\, a^*(z) a(z),$$
$$f(z) \longmapsto -a^*(z)^2 a(z),$$

where

- $x(z) = \sum x_n z^{-n-1}$, $x_n = x t^n$, $x \in \mathfrak{g}$,
- $a^*(z) = \sum a^*_n z^{-n}$, $a^*_n = y_{-n}$,
- $a(z) = \sum a_n z^{-n-1}$, $a_n = \partial_{y_n}$.

However, these op's do NOT act on the Fock mod. of Weyl alg.:

$M = \mathbb{C}[a^*_0, a^*_{-1}, \cdots, a_{-1}, a_{-2}, \cdots] |0\rangle$.

} "renormalize"

$$\hat{\rho}(e(z)) = a(z),$$
$$\hat{\rho}(h(z)) = -2 :a^*(z) a(z):,$$
$$\hat{\rho}(f(z)) = - :a^*(z)^2 a(z): - 2\partial a^*(z)$$

defines an $\hat{\mathfrak{sl}}_2$-mod. str. on M at level $-h^\vee = -2$ ([Wakimoto 86]).

In general,

Thm. ([Feigin–Frenkel 88]).

$\hat{\rho} : \hat{\mathfrak{g}} \curvearrowright M$: at level $-h^\vee$,

$\hat{\rho}^k : \hat{\mathfrak{g}} \curvearrowright M \otimes \mathcal{F}^{k+h^\vee}$: at level k,

where

$\mathcal{F}^{k+h^\vee}$: Fock mod. of Heisenberg Lie algebra $\hat{\mathfrak{h}}$ on which $\mathbb{1} = k + h^\vee$.

§ For basic classical Lie superalgebras

$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$: Lie superalgebra

Exm.

$\mathfrak{gl}_{m|n} = (\mathfrak{gl}_{m|n})_{\bar{0}} \oplus (\mathfrak{gl}_{m|n})_{\bar{1}}$,

$(\mathfrak{gl}_{m|n})_{\bar{0}} = \left\{ \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \right\}$,

$(\mathfrak{gl}_{m|n})_{\bar{1}} = \left\{ \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \right\}$.

Def.

$\mathfrak{g}$: basic classical simple Lie superalgebra

$:\Leftrightarrow$
- simple,
- $\mathfrak{g}_{\bar{0}}$: reductive,
- $\mathfrak{g}$ has a nondeg. inv. even supersym. bilinear form.

$\Vert$

Killing form

Exm.

$\mathfrak{osp}(1|2) = \{ X \in \mathfrak{gl}_{1|2} \mid {}^{st}XB + BX = 0 \}$,

where

$$B = \left[\begin{array}{c|cc} 1 & & \\ \hline & & -1 \\ & 1 & \end{array}\right].$$

Def.

$\mathfrak{g}$: basic classical simple Lie superalg.

$(,)$: a fixed nondeg. inv. even supersym. bilinear form.

We define an affine Lie superalgebra by

$$\hat{\mathfrak{g}} = \mathfrak{g}((t)) \oplus \mathbb{C}K$$

with a Lie bracket given by

$$[x_m, y_n] = [x, y]_{m+n} + m\,\delta_{m+n,0}(x, y)K,$$

K : central,

where $x_m = xt^m$, $x \in \mathfrak{g}$.

For a fixed $(,)$ on $\mathfrak{g}$, we define $h^\vee \in \mathbb{C}$ to be

$$2h^\vee(,) = (\text{Killing form})$$

Thm. ([Genra – I. – Nakatsuka])

$\hat{\rho} : \hat{\mathfrak{g}} \curvearrowright M$: at level $-h^\vee$,

$\hat{\rho}^k : \hat{\mathfrak{g}} \curvearrowright M \otimes \pi^{k+h^\vee}$: at level k.

ass. with $N+$: super

$M = (a_\alpha, a^*_\alpha)$

In the vertex algebra language, this claim is rewritten as :

$\hat{\rho}^k : V^k(\mathfrak{g}) \longrightarrow M \otimes \pi^{k+h^\vee}$: hom. of vertex alg's

univ. aff vertex superalg.

Thm. ([Genra – I. – Nakatsuka])

For any $k \in \mathbb{C}$, $\hat{\rho}^k$ is an injective.

We also have screening operators.

Thm. ([Genra – I. – Nakatsuka]).

$$V^k(\mathfrak{g}) \simeq \operatorname{Im} \hat{\rho}^k = \bigcap_{i=1}^{\dim \mathfrak{h}} \operatorname{Ker} S_i .$$

§ For $\mathcal{N}=1$ SUSY affine Lie superalgebras

Def.

We define an $\mathcal{N}=1$ SUSY affine Lie superalgebra by

$$\hat{\mathfrak{g}}_{\mathcal{N}=1} = \mathfrak{g}((t))[\omega] \oplus \mathbb{C}K,$$

t : even variable,

ω : odd variable,

with a Lie bracket given by

$$[x_m, y_n] = [x, y]_{m+n} + m \delta_{m+n,0} (x, y) K,$$

$$[\bar{x}_{m+\frac{1}{2}}, y_n] = \overline{[x, y]}_{m+n+\frac{1}{2}},$$

$$[\bar{x}_{m+\frac{1}{2}}, \bar{y}_{n+\frac{1}{2}}] = (-1)^{|x|} \delta_{m+n+1,0} (x, y) K,$$

K : central,

where

$$x_m = x t^m, \quad \bar{x}_{m+\frac{1}{2}} = x t^m \omega, \quad x \in \mathfrak{g}.$$

$\mathfrak{g} \longrightarrow \mathrm{Vect}(U), \quad U \subset \mathbb{B}|G.$

{ "super loop"

$\mathfrak{g}((t))[\omega] \longrightarrow \mathrm{Vect}(\mathbb{L}^{\mathcal{N}=1} U)$

} "renormalize"

Thm. ([I.1])

$\hat{\rho}$: $\hat{\mathfrak{g}}_{N=1} \curvearrowright M_{\beta\gamma\text{-}bc}$: at level 0.

$\hat{\rho}^k$: $\hat{\mathfrak{g}}_{N=1} \curvearrowright M_{\beta\gamma\text{-}bc} \otimes \Pi^k$: at level k,

where

(i) $M_{\beta\gamma\text{-}bc} = M \otimes \mathcal{F}_{ch}$,

(ii) $\Pi^k = \pi^k \otimes \mathcal{F}^k_{neu}$.

In the vertex algebra language, this claim is rewritten as

$$\hat{\rho}^k : V^k_{N=1}(\mathfrak{g}) \longrightarrow M_{\beta\gamma\text{-}bc} \otimes \Pi^k.$$

univ. $N=1$ SUSY aff VA.

In [I.1], we observe the new phenomenon:

Thm. ([I.])

$k \neq 0 \Rightarrow \hat{\rho}^k$: injective,

$k = 0 \Rightarrow \hat{\rho}^k$: NOT injective.

We have candidates for screening operators:

Thm. ([I.1])

$$\mathrm{Im}\,\hat{\rho}^k \subset \bigcap_{z=1}^{\dim \mathfrak{h}} \ker S_z.$$

<Thank you!>

