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Quantifying Measurement Objectivity: A Retrodictive Approach

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When Does an Outcome Reveal a Property?

- **Classical**: the outcome is a function of the system's state → it reads off a value already there
- **Quantum**: the Bell and Kochen–Specker theorems limit locally causal and noncontextual accounts of pre-existing values

  → but not for every state and measurement: classical situations remain inside quantum theory

For a given measurement $\mathbf{P}$ and prior state γ:

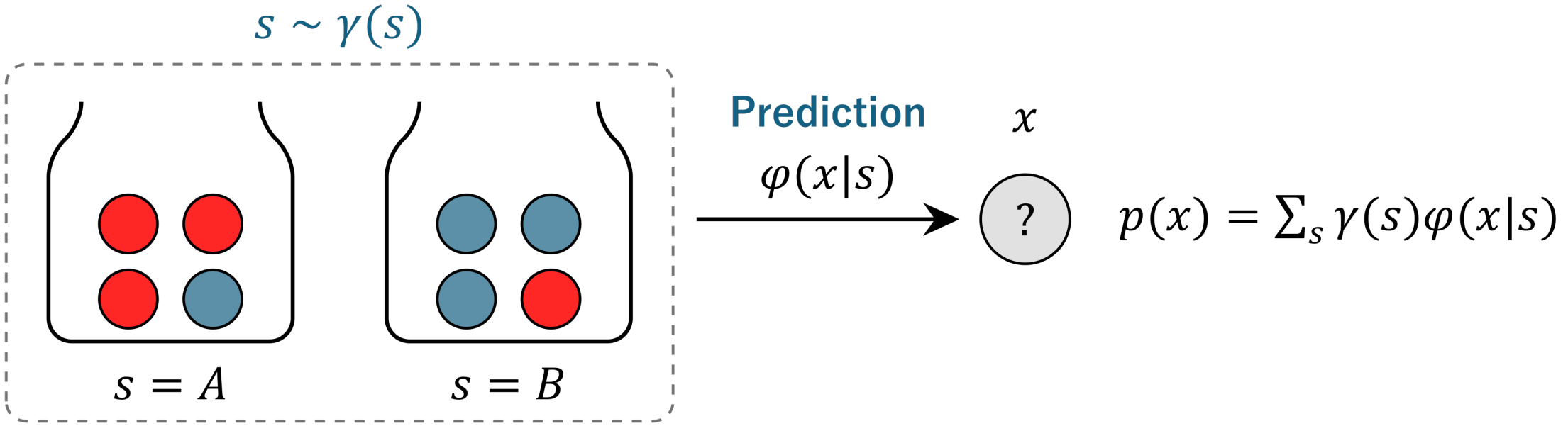
① Can the outcome be read as revealing a pre-existing property?

② If not, which functions of the outcome still can?

③ How much non-objectivity remains?

Prediction and Retrodiction

Urn s chosen with prior $\gamma(s)$; ball color x drawn with $\varphi(x|s)$



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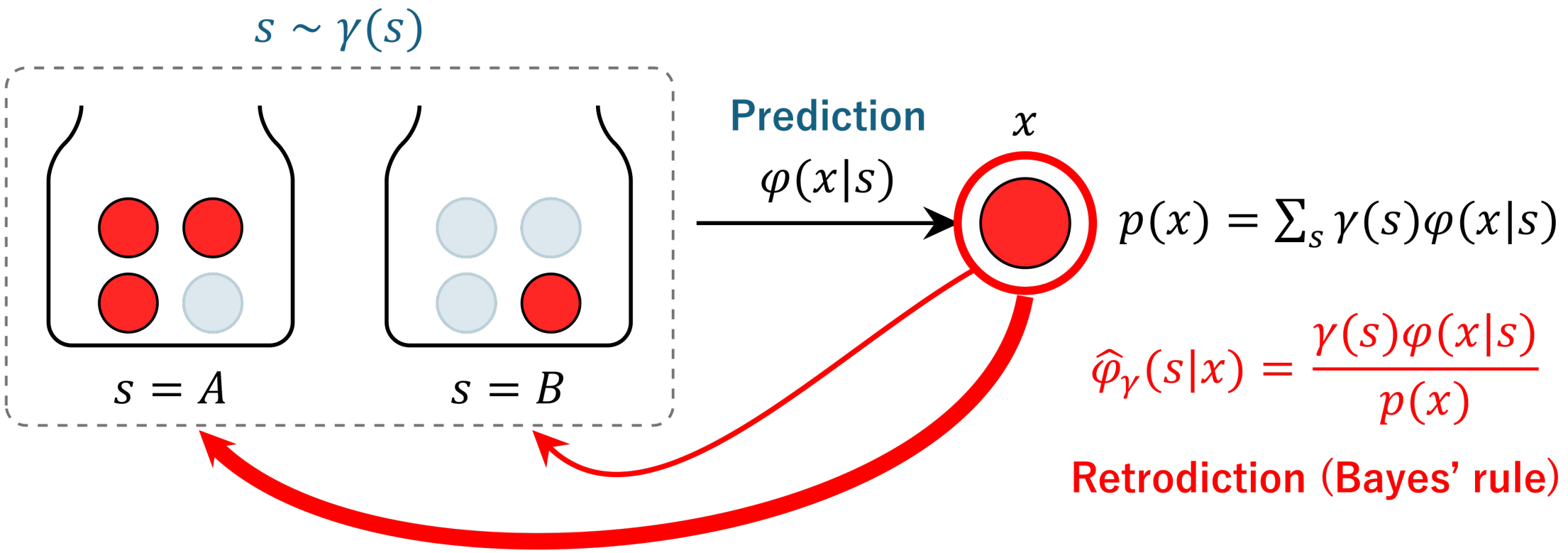


Quantum Setting

- **Prior**: density operator γ of the input system Q, faithful $(\gamma > 0)$
- **Measurement**: POVM $\mathbf{P} = \{P_x\}_{x\in\mathcal{X}}$, $P_x \geq 0$, $\sum_x P_x = \mathbb{1}$; Born rule $p_x = \mathrm{Tr}\{\gamma P_x\} > 0$
- **Evidence**: a distribution τ on the outcomes, e.g. $\tau(x) = \delta_{x\bar{x}}$ after observing $\bar{x}$

Process and input together: one **joint state (Choi)** of the outcome X and the input Q

Forward (prior and measurement): $\gamma_{XQ} = \sum_x |x\rangle\langle x|_X \otimes \sqrt{\gamma}\, P_x \sqrt{\gamma}$

Backward (evidence, backward channel $x \mapsto \sigma_x$): $\sigma_{XQ} = \sum_x \tau(x)\, |x\rangle\langle x|_X \otimes \sigma_x$

Only the states σ_x are free

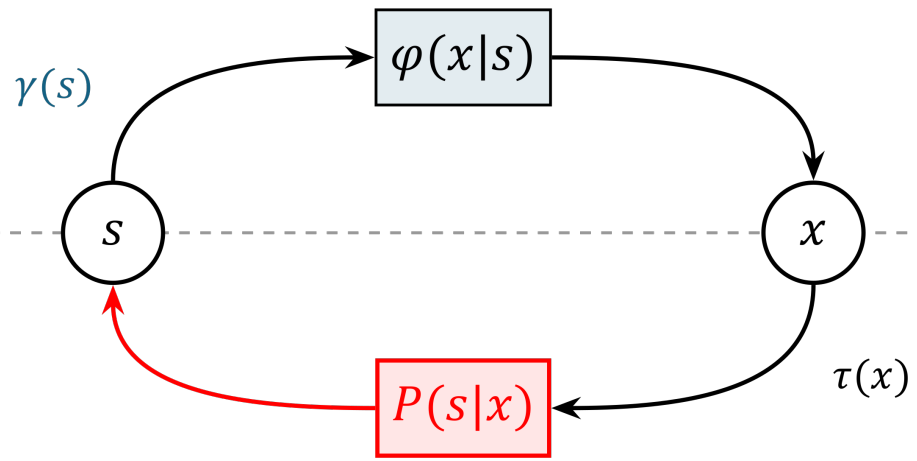
Classical analogue: $P_{\mathrm{fwd}}(s,x) = \gamma(s)\,\varphi(x|s)$ and $P(s,x) = \tau(x)\,P(s|x)$

Minimum Change: Forward vs. Backward

Classical

$$P_{\text{fwd}}(s,x) = \gamma(s)\,\varphi(x|s)$$

Forward
reference



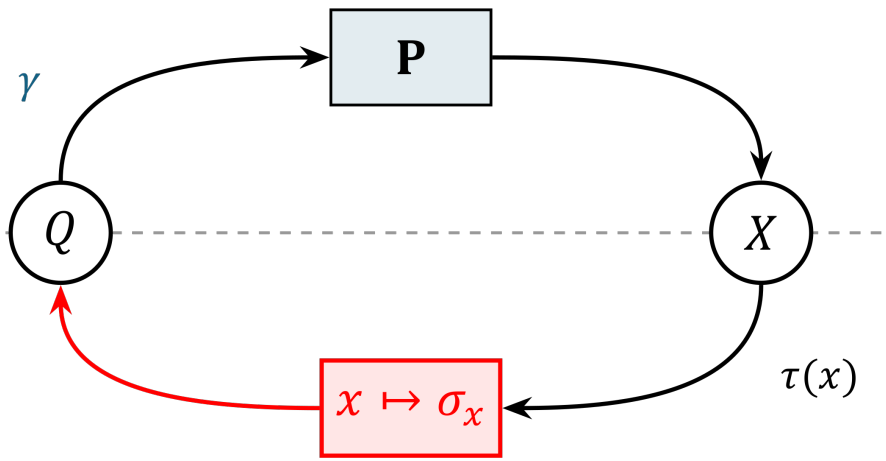
Backward
update

$$P(s,x) = \tau(x)\,P(s|x)$$

$$\min D(P\|P_{\text{fwd}})$$

Quantum

$$\gamma_{XQ} = \sum_x |x\rangle\langle x| \otimes \sqrt{\gamma} P_x \sqrt{\gamma}$$



$$\sigma_{XQ} = \sum_x \tau(x)|x\rangle\langle x| \otimes \sigma_x$$

$$\min \mathbb{D}(\sigma_{XQ}\|\gamma_{XQ})$$

change as little as possible

Quantum Retrodiction

For every $\mathbb{D}$ in a wide family of quantum relative entropies, the minimizer is unique and the same:

(Umegaki, Belavkin–Staszewski; Petz, sandwiched, and geometric Rényi)

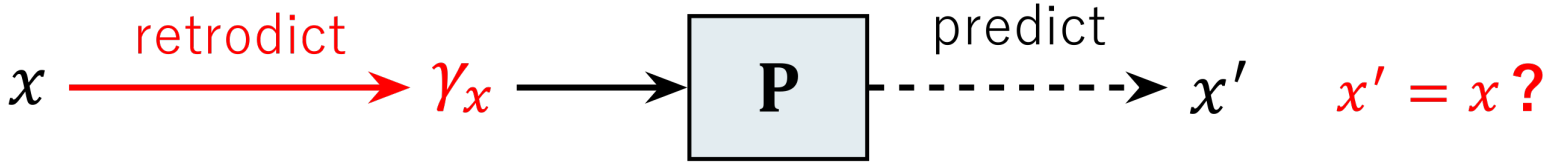
$$\hat{\varphi}_\gamma(s|x) = \frac{\gamma(s)\,\varphi(x|s)}{p(x)} \quad \longrightarrow \quad \sigma_x = \gamma_x = \frac{\sqrt{\gamma}\,P_x\sqrt{\gamma}}{p_x}$$

J. Kuang, K. Torii, and F. Buscemi, Quantum Sci. Technol. 11, 025049 (2026)

- Minimum value $= \mathbb{D}(\tau\|p)$: the quantum part adds no change beyond the evidence
- γ_x describes the system before the measurement, given x; not a post-measurement state

When Is a Measurement Objective?

Retrodict and predict with the same POVM $\mathbf{P} = \{P_x\}$, faithful prior $\gamma > 0$:



$$T_{xx'} := p_x \mathrm{Tr}\{\gamma_x P_{x'}\} = \mathrm{Tr}\{\sqrt{\gamma}\, P_x \sqrt{\gamma}\, P_{x'}\}, \qquad \gamma_x = \frac{\sqrt{\gamma}\, P_x \sqrt{\gamma}}{p_x}$$

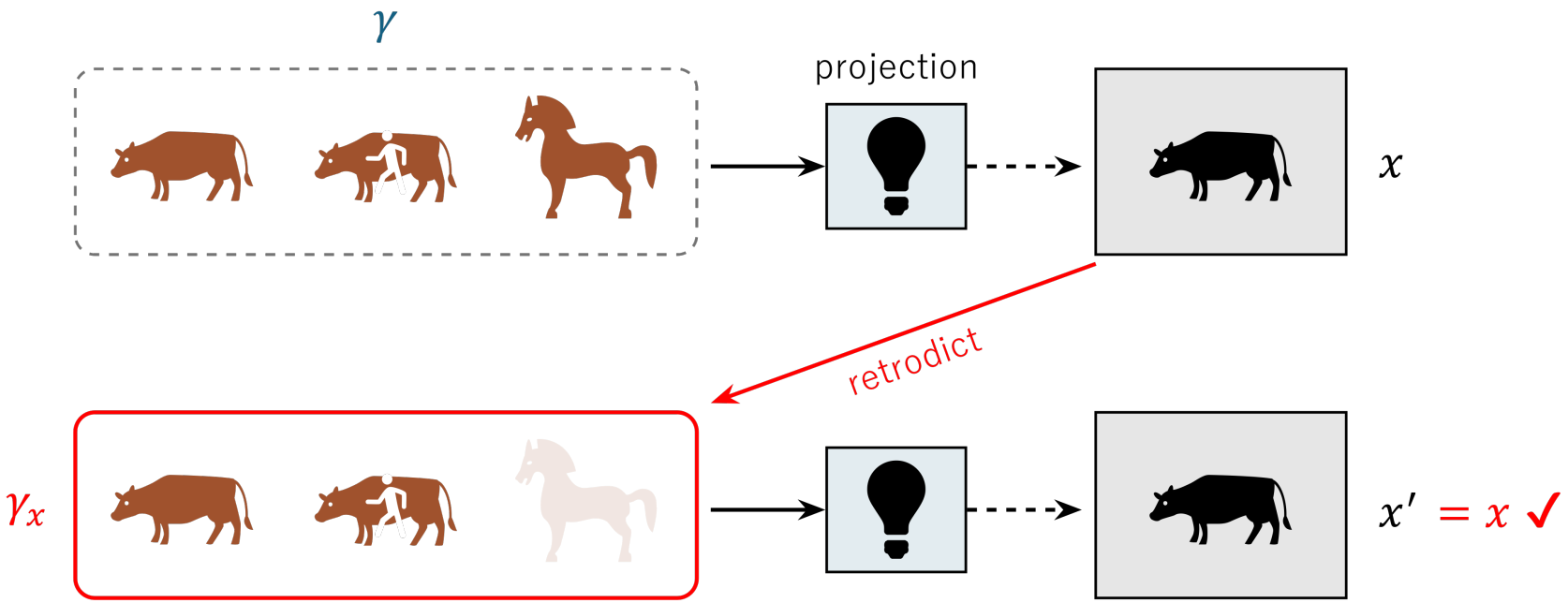
Objective relative to γ: $T_{xx'} = p_x\, \delta_{xx'} \quad \Leftrightarrow \quad \mathrm{Tr}\{\gamma_x P_{x'}\} = \delta_{xx'}$

→ e.g. sharp (PVM) and commuting with the prior ⇒ objective

Non-objectivity: $\mathbf{N}_{xx'} := p_x\, \delta_{xx'} - T_{xx'}$

Objectivity: A Classical Picture

Objects in the dark, observed only through their shadow



Projecting the retrodicted objects again: the same silhouette with certainty

→ x reveals a property already present, relative to this prior and this projection

It does not identify the object: a cow, or a person in a cow costume

Non-Objectivity Tensor

Outcome feature: a real function $v: \mathcal{X} \to \mathbb{R}$ of the label, with operators

$$O_v := \sum_x v_x \, P_x, \qquad O_{v^2} := \sum_x v_x^2 \, P_x$$

Non-objectivity of v (**N** from p. 8)

$$v^\top \mathbf{N} v = \frac{1}{2}\Big[\underbrace{\mathrm{Tr}\{\gamma O_{v^2}\}}_{\mathbb{E}[v(X)^2]} - 2\,\underbrace{\mathrm{Tr}\{\sqrt{\gamma}\, O_v \sqrt{\gamma}\, O_v\}}_{\mathbb{E}[v(X)\,v(X')]} + \underbrace{\mathrm{Tr}\{\gamma O_{v^2}\}}_{\mathbb{E}[v(X')^2]}\Big] = \frac{1}{2}\,\mathbb{E}[(v(X) - v(X'))^2]$$

- $\mathbf{N} \geq 0$; feature v is objective $\Leftrightarrow v \in \ker \mathbf{N}$

Unsharpness and Asymmetry

Theorem 1 (decomposition)

$$\mathbf{N} = \mathbf{U} + \mathbf{A}, \quad \mathbf{U}, \mathbf{A} \geq 0$$
$$\mathbf{U} = 0 \Leftrightarrow \mathbf{P} \text{ is a PVM}$$
$$\mathbf{A} = 0 \Leftrightarrow [\gamma, P_x] = 0 \ \ \forall x$$

Unsharpness: $v^{\mathsf{T}}\mathbf{U}v = \mathrm{Tr}\{\gamma O_{v^2}\} - \mathrm{Tr}\{\gamma O_v^2\}$

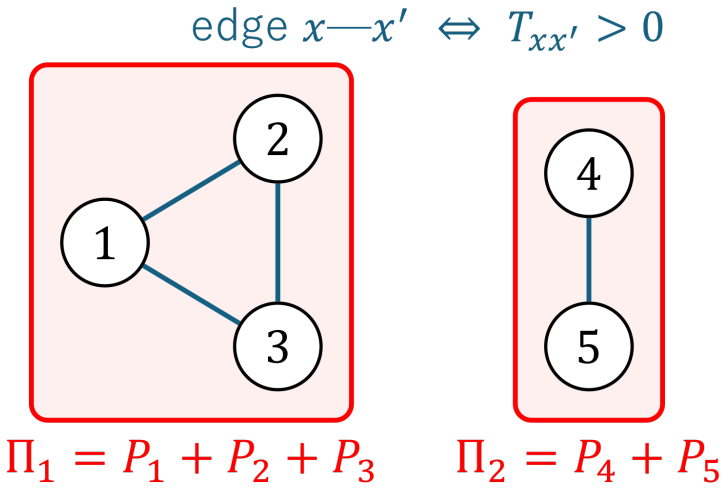
Asymmetry: $v^{\mathsf{T}}\mathbf{A}v = \mathrm{Tr}\{\gamma O_v^2\} - \mathrm{Tr}\{\sqrt{\gamma}\, O_v \sqrt{\gamma}\, O_v\} = I_{\mathrm{WY}}(\gamma, O_v)$

I_{WY}: Wigner–Yanase skew information

→ objective ($\mathbf{N} = 0$) ⇔ sharp (PVM) and commuting with the prior

*the easy case is the only one

Objective Features and Coarse-Graining



- Connect $x \neq x'$ when $T_{xx'} > 0$
- Objective features = functions constant on each connected component

N: weighted graph Laplacian

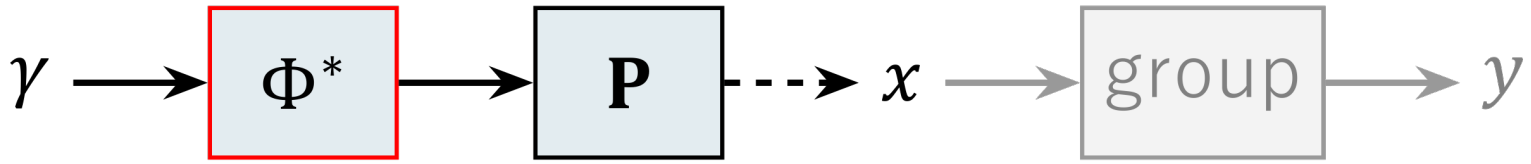
Corollary 1.1

$$\ker \mathbf{N} = \mathrm{span}_{\mathbb{R}}\{\mathbf{1}_{\mathcal{X}_y}\}_y, \qquad \Pi_y := \sum_{x \in \mathcal{X}_y} P_x$$

→ Π is the unique finest objective coarse-graining (MPPP)

T. Nagasawa et al., Rep. Prog. Phys. 88, 117601 (2025)

Can We Make a Measurement More Objective?



before:
quantum preprocessing

after:
group outcomes (p. 12)

$$\mathbf{P} \mapsto \Phi(\mathbf{P}) = \{\Phi(P_x)\}_x, \qquad \Phi \text{ unital CP}, \quad \Phi^*(\gamma) = \gamma$$

Theorem 2

$$\mathbf{U}' \geq \mathbf{U}, \qquad \mathbf{N}' \geq \mathbf{N}$$

After: grouping outcomes can buy objectivity, at the cost of resolution

Before: prior-preserving preprocessing cannot create objectivity

Conclusions and Outlook

Objectivity relative to a prior = retrodictive certainty

① Objective (T diagonal) ⇔ sharp (PVM) and commuting with the prior $(\mathbf{N} = \mathbf{U} + \mathbf{A})$

② Objective features $= \ker \mathbf{N}$: the unique finest objective coarse-graining (MPPP)

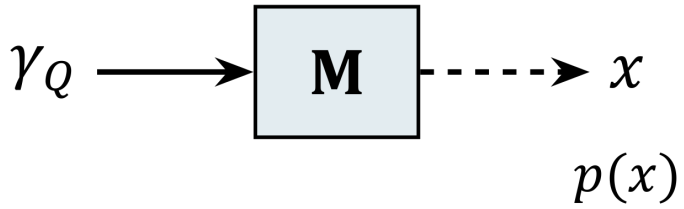
③ Non-objectivity $v^T \mathbf{N} v$ of a feature v cannot be reduced by prior-preserving preprocessing: $\mathbf{N}' \geq \mathbf{N}$, $\mathbf{A} - \mathbf{A}' \leq \mathbf{U}' - \mathbf{U}$

Outlook: stability of approximate objectivity

J. Kuang, T. Nagasawa, K. Torii, and F. Buscemi, Quantifying measurement objectivity: a retrodictive approach, under second round of review (2026)
J. Kuang, K. Torii, and F. Buscemi, Quantum measurement retrodiction and entropic uncertainty relations, Quantum Sci. Technol. 11, 025049 (2026)

Appendix

A1. Setting of Minimum-Change Retrodiction



- Prior: faithful density operator $\gamma_Q > 0$ on a finite-dimensional $\mathcal{H}_Q$
- Measurement: finite POVM $\mathbf{M} = \{M_x\}_{x\in\mathcal{X}}$, $M_x \geq 0$, $\sum_x M_x = \mathbb{1}$, no zero effects
- Born probabilities: $p(x) = \mathrm{Tr}\{\gamma_Q M_x\} > 0,\ \ x \in \mathcal{X} = \{1, \ldots, n\}$
- Forward process as a classical–quantum state (Choi); block x has trace $p(x)$:

$$\gamma_{XQ} := \sum_x |x\rangle\langle x|_X \otimes \sqrt{\gamma_Q}\, M_x \sqrt{\gamma_Q} = \begin{pmatrix} \sqrt{\gamma_Q}\, M_1 \sqrt{\gamma_Q} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\gamma_Q}\, M_n \sqrt{\gamma_Q} \end{pmatrix}$$

A2. The Minimum-Change Problem

- New evidence: a target distribution τ on $\mathcal{X}$
- Admissible backward states (only the conditional states are free):

$$\mathcal{A}_\tau := \left\{ \sigma_{XQ} = \sum_x \tau(x) |x\rangle\langle x|_X \otimes \sigma_{Q|x} \;:\; \sigma_{Q|x} \in \mathcal{S}(\mathcal{H}_Q) \right\}$$

$$\min_{\sigma_{XQ} \in \mathcal{A}_\tau} \mathbb{D}(\sigma_{XQ} \| \gamma_{XQ})$$

- $\mathbb{D}$: Umegaki, Belavkin–Staszewski, Petz / sandwiched / geometric Rényi

Does the update depend on the choice of divergence?

A3. Divergence Independence

Theorem. For every listed $\mathbb{D}$ and every target distribution τ,

$$\min_{\sigma_{XQ}\in\mathcal{A}_\tau} \mathbb{D}(\sigma_{XQ}\|\gamma_{XQ}) = \mathbb{D}(\tau\|p)$$

and the unique minimizer is

$$\hat{\gamma}_{XQ} = \sum_x \tau(x)|x\rangle\langle x|_X \otimes \hat{\gamma}_{Q|x}, \qquad \hat{\gamma}_{Q|x} = \frac{\sqrt{\gamma_Q}\, M_x \sqrt{\gamma_Q}}{p(x)}$$

→ Minimum change selects the Bayesian inverse (measurement case of the Petz map), for all these divergences

Trace distance: same minimizer, but not necessarily unique (A5)

A4. Proof Sketch of Divergence Independence

1. Data processing under the partial trace over Q gives a lower bound:

$$\mathbb{D}(\sigma_{XQ}\|\gamma_{XQ}) \geq \mathbb{D}(\mathrm{Tr}_Q\{\sigma_{XQ}\}\|\mathrm{Tr}_Q\{\gamma_{XQ}\}) = \mathbb{D}(\tau\|p)$$

2. Block decomposition (example: Umegaki relative entropy):

$$D(\sigma_{XQ}\|\gamma_{XQ}) = D(\tau\|p) + \sum_{x:\tau(x)>0} \tau(x)\, D(\,\sigma_{Q|x}\|\hat{\gamma}_{Q|x})$$

3. Nonnegativity and faithfulness: equality iff $\sigma_{Q|x} = \hat{\gamma}_{Q|x}$ whenever $\tau(x) > 0$

Rényi families: same argument through their common exponent (Khatri–Lami–Wilde, Secs. 7.2, 7.4–7.7)

A5. Trace Distance

$$\min_{\sigma_{XQ} \in \mathcal{A}_\tau} \frac{1}{2} \|\sigma_{XQ} - \gamma_{XQ}\|_1 = \frac{1}{2} \sum_x |\tau(x) - p(x)|$$

- The Bayesian inverse attains the minimum
- Other minimizers may exist: $\|A\|_1 \geq |\mathrm{Tr}\{A\}|$ is tight for every block that is positive or negative semidefinite, so the conditional states need not be fixed

A6. Petz Map and the Bayesian Inverse

Measurement channel of $\mathbf{M}$:

$$\mathcal{E}(\rho) = \sum_x \mathrm{Tr}\{M_x \rho\} |x\rangle\langle x|_X$$

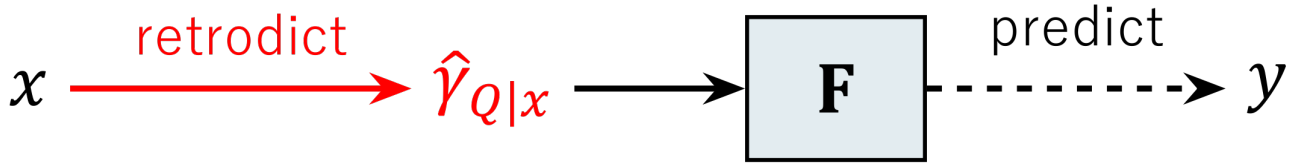
Petz recovery map of $\mathcal{E}$ with respect to γ:

$$\mathcal{P}_{\gamma,\mathcal{E}}(Z) = \sqrt{\gamma}\, \mathcal{E}^\dagger\left(\mathcal{E}(\gamma)^{-1/2} Z\, \mathcal{E}(\gamma)^{-1/2}\right)\sqrt{\gamma}, \qquad \mathcal{P}_{\gamma,\mathcal{E}}(|x\rangle\langle x|) = \hat{\gamma}_{Q|x}$$

The Bayesian inverse is the measurement case of the Petz map; minimum change selects it without assuming a fidelity criterion

A7. Joint Law for Two Measurements

Retrodict with $\mathbf{M}$, then predict a second POVM $\mathbf{F} = \{F_y\}_{y \in \mathcal{Y}}$:



$$\Pr\{y \leftarrow x\} := p(x)\mathrm{Tr}\{F_y \hat{\gamma}_{Q|x}\} = \mathrm{Tr}\{F_y \sqrt{\gamma_Q}\, M_x \sqrt{\gamma_Q}\}$$

- Symmetric under $\mathbf{M} \leftrightarrow \mathbf{F}$
- Marginals: the Born distributions $p(x)$ and $q(y)$
- Realized by a sequential measurement at the prior (A8)
- $\mathbf{F} \neq \mathbf{M}$: Useful for discussing compatibility problems

→ Same measurement in both roles ($\mathbf{F} = \mathbf{M}$): objectivity, p. 8

A8. Sequential Realization of the Joint Law

$$L_{\gamma,x} := \left(\sqrt{\gamma_Q} M_x \sqrt{\gamma_Q}\right)^{1/2} \gamma_Q^{-1/2}: \qquad L_{\gamma,x}^\dagger L_{\gamma,x} = M_x, \qquad L_{\gamma,x} \gamma_Q L_{\gamma,x}^\dagger = \sqrt{\gamma_Q} M_x \sqrt{\gamma_Q}$$

$$\widetilde{\mathcal{J}}^{\mathbf{M}\to\mathbf{F}}(Z) = \sum_{x,y} \sqrt{F_y}\, L_{\gamma,x} Z L_{\gamma,x}^\dagger \sqrt{F_y} \otimes |x\rangle\langle x|_X \otimes |y\rangle\langle y|_Y$$

- At the prior, the first stage outputs the unnormalized Bayesian inverse $\sqrt{\gamma_Q} M_x \sqrt{\gamma_Q}$
- A subsequent measurement of $\mathbf{F}$ records (x, y) with probability $\Pr\{y \leftarrow x\}$
- The joint law is a realization at the specified prior, not an interpretation of a post-measurement state

A9. Retrodictive Assignment and Physical States

- $\hat{\gamma}_{Q|x}$ is assigned to the system before the measurement, given x; it depends only on $(\gamma_Q, \mathbf{M})$
- The post-measurement state depends on the instrument that implements $\mathbf{M}$
- The objectivity criterion concerns the POVM and the prior, not repeatability of an instrument
- This talk assumes a faithful (full-rank) prior throughout

A10. Other Approaches to Objectivity

- Quantum Darwinism: redundant records of the system in environment fragments
- Spectrum broadcast structures: distinct alternatives recorded in distinguishable environmental states
- Here: a single POVM and a prior; agreement among observers would require further systems and their correlations
- Related: prior-compatible projective post-processing (Nagasawa et al., 2025)

A11. Best Feature and HGR Maximum Correlation

Least non-objectivity over zero-mean, unit-variance features ($|\mathcal{X}| \geq 2$):

$$\nu_\gamma(\mathbf{P}) := \min_{\mathbb{E}[v(X)]=0, \mathbb{E}[v(X)^2]=1} v^\top \mathbf{N} v$$

- $\nu_\gamma(\mathbf{P}) = 0 \Leftrightarrow$ the MPPP has at least two outcomes
- $1 - \nu_\gamma(\mathbf{P}) = \xi_{\mathrm{HGR}}(X, X')$: HGR maximum correlation of the retrodictive joint law
- Prior-preserving preprocessing: $\nu_\gamma(\Phi(\mathbf{P})) \geq \nu_\gamma(\mathbf{P})$
- Continuous in $(\mathbf{P}, \gamma)$ while all $p_x > 0$; relation to nearby exactly objective coarse-grainings is open

A12. Asymmetry–Unsharpness Trade-off

Prior-preserving Φ, also covariant under the modular group of γ:

$$\mathbf{P} \mapsto \Phi(\mathbf{P}) = \{\Phi(P_x)\}_x, \qquad \Phi \text{ unital CP}, \quad \Phi^*(\gamma) = \gamma$$

Corollary 2.1

$$\mathbf{A}' \leq \mathbf{A}, \qquad 0 \leq \mathbf{A} - \mathbf{A}' \leq \mathbf{U}' - \mathbf{U}$$

→ less asymmetry costs at least as much additional unsharpness