Spectral Sequences Meet Topological Phases

Domain Wall Decorations and Interacting Topological Superconductors

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Q.-R. Wang, S.-Q. Ning, M. Cheng, arXiv:2104.13233

S.-Q. Ning, X.-Y. Ren, Q.-R. Wang, Y. Qi, Z.-C. Gu, arXiv:2512.25069



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Q.-R. Wang, S.-Q. Ning, M. Cheng, arXiv:2104.13233
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What is a symmetry-protected topological phase?

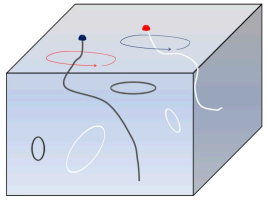
Symmetry-protected topological (SPT) phases

- **Short-range entangled:** a unique gapped ground state, no anyons.
- **Nontrivial only with symmetry:** break G and it connects to a product state.
- **Exotic boundaries:** gapless, symmetry-breaking, or a surface topological order.
- **Interactions** allow new phases with no free-fermion counterpart.

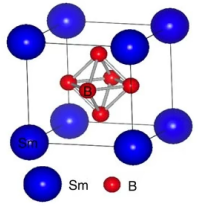
Examples

Haldane chain





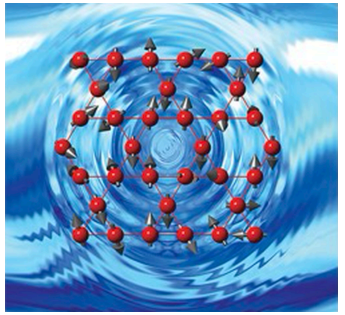
3D topological paramagnet



interacting candidate: SmB_6

Contrast: intrinsic topological order

- Anyon excitations, long-range entanglement
- No symmetry needed; described by a category
- Add symmetry ⇒ symmetry-**enriched** phases, e.g. for $SO(3)$ an anyon can carry spin-½



quantum spin liquid



FQHE

Motivation

Q1 Is there a unified mathematical structure behind the classification of symmetry-protected topological (SPT) phases, with a clear physical picture?

Q2 Can it be made computable in practice, for interacting fermions and for crystalline symmetries, e.g. all 230 space groups?

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Main message

The decorated domain wall construction is organized by the Atiyah–Hirzebruch spectral sequence.

Bosonic SPT

The spectral sequence at cochain level

Cochain-level formulas · LSM theorems · SET gauge theories

arXiv:2104.13233

Fermionic SPT

Interacting crystalline topological superconductors

Four-layer decoration data · all 230 space groups

arXiv:2512.25069

Spectral sequences for topological phases: a short history

2014

Decorated domain walls

Chen, Lu, Vishwanath

physical picture of SPTs

2017

SPTs as generalized cohomology

Xiong; Gaiotto–Johnson-Freyd

Kitaev's proposal made precise

2018

Real-space recipes, defect networks

Song–Fang–Qi; Else–Thorngren

all 230 space groups (bosons)

2021

LHS at cochain level

Wang, Ning, Cheng

LSM theorems, SET theories

2025

Interacting crystalline TSCs

Ning, Ren, Wang, Qi, Gu

four layers, 230 space groups

2017

Block states & crystalline equivalence

Song et al.; Thorngren–Else

crystalline ↔ internal symmetry

2018

AHSS for crystalline SPTs

Shiozaki, Xiong, Gomi

layers, higher-order SPTs, LSM

2018

Fermionic decorations

Wang, Gu

Kitaev-chain and fermion layers

2022

AHSS in band topology

Shiozaki, Sato, Gomi

free fermions, 230 space groups

● this talk

Decorated domain walls or defects

An SPT ground state is a **superposition of all domain-wall configurations**, each wall dressed with a lower-dimensional SPT (invertible phases). Letting the decorated walls fluctuate restores the symmetry; the decoration fixes the phase.

i Decorate

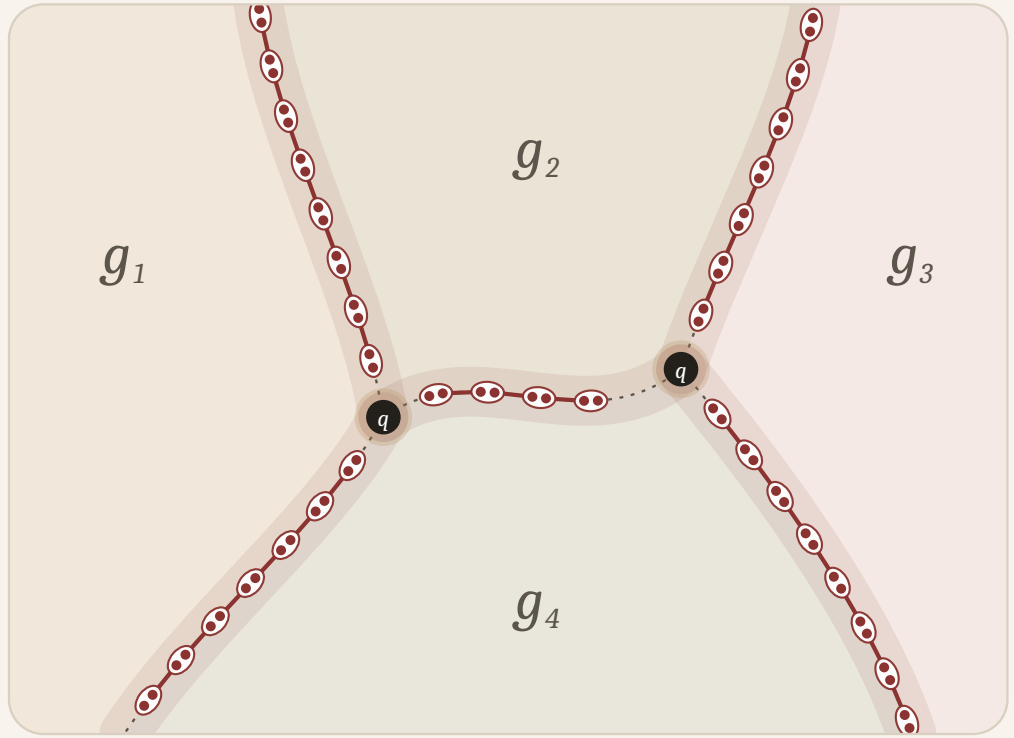
Place lower-dimensional SPT states (invertible phases) on the walls and on their junctions, e.g. Haldane chains on 1D walls, charges at 0D junctions.

ii Constrain

Decorations must fuse consistently where walls meet; a failure of consistency is an **obstruction**.

iii Classify

Consistent decoration patterns, up to trivial ones, give the SPT phases.



Four domains: each wall carries a Haldane chain, each junction a charge.

Example: the $\mathbb{Z}_2 \times \mathbb{Z}_2$ cluster chain

On every domain wall of one $\mathbb{Z}_2$, decorate a 0D charge of the other $\mathbb{Z}_2$.

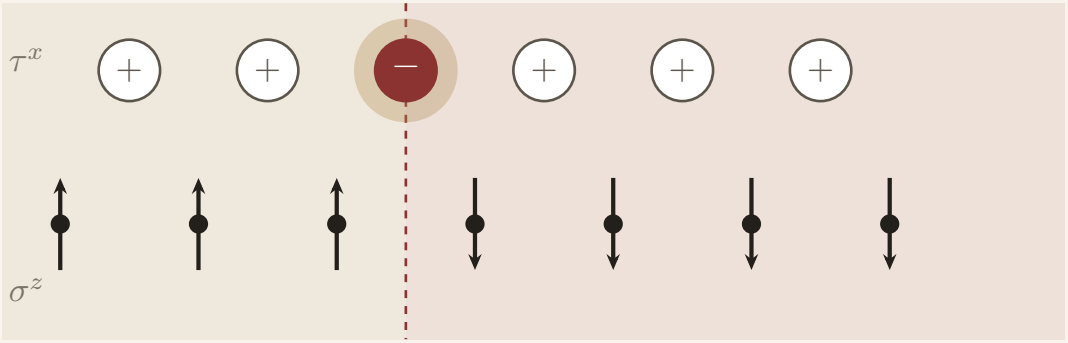
Wave function

$$|\psi\rangle = \sum_{\{\sigma\}} |\{\sigma\}\rangle \otimes \left|\{\tau^x_{i+\frac{1}{2}} = \sigma^z_i \sigma^z_{i+1}\}\right\rangle$$

Parent Hamiltonian

$$H = -\lambda \sum_i \left(\sigma^z_i \sigma^z_{i+1} \tau^x_{i+\frac{1}{2}} + \tau^z_{i-\frac{1}{2}} \tau^z_{i+\frac{1}{2}} \sigma^x_i\right)$$

Each term fixes $\tau^x = -1$ exactly where neighbouring σ spins are anti-parallel: the τ charge sits on the σ domain wall. Summing over all σ configurations restores both symmetries.



σ domain wall carries τ charge

At the edge

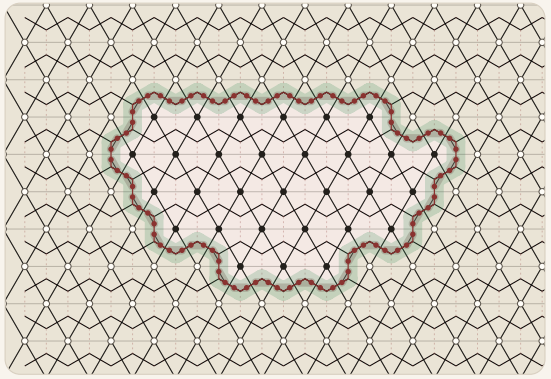
At an open end the two symmetries $\mathbb{Z}_2^\sigma$ and $\mathbb{Z}_2^\tau$ anticommute: a projective representation, like a spin-½.

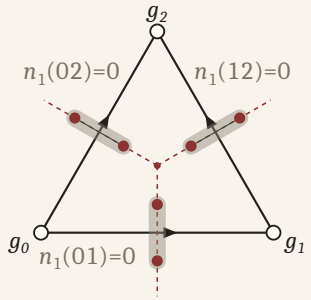
Both the wavefunction and the Hamiltonian are constructed from the domain wall decoration.

Chen, Lu, Vishwanath, Nat. Commun. 5, 3507 (2014)

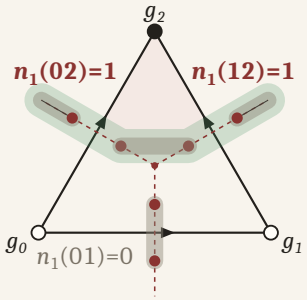
Kitaev-chain decoration in 2D and 3D

2D $\mathbb{Z}_2 \times \mathbb{Z}_2^f$: Kitaev chain on the domain wall

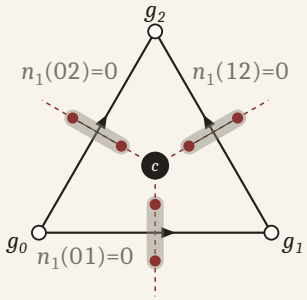




vacuum

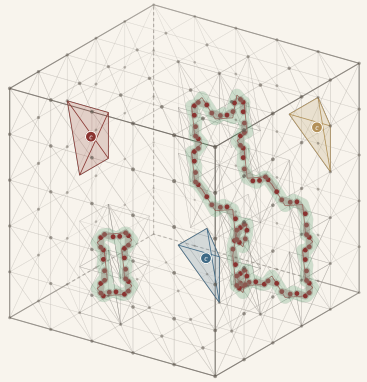


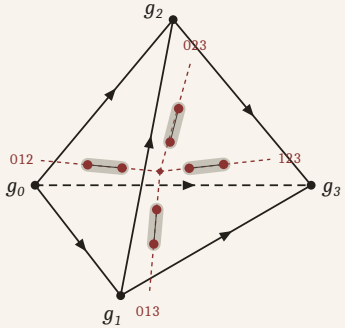
chain passes through



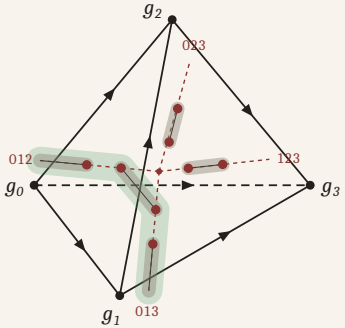
$n_2(012) = 1$: fermion c

3D: Kitaev chains on lines, complex fermions at points

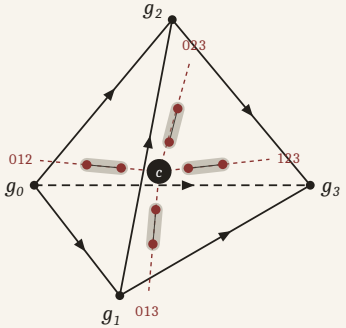




vacuum



$n_2(012) = n_2(013) = 1$



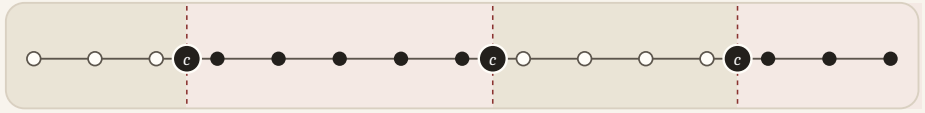
$n_3(0123) = 1$: fermion c

The decoration rule can be specified **microscopically** and **locally**.

Construction: Wang, Gu, PRX 8, 011055 (2018); PRX 10, 031055 (2020); arXiv:2512.25069

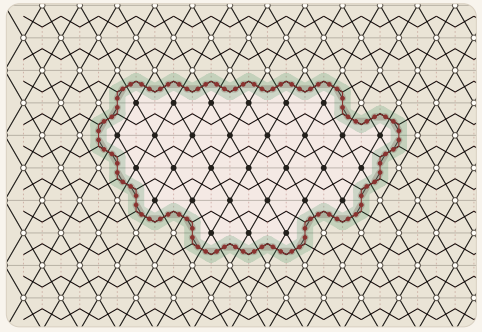
Consistency conditions

1D



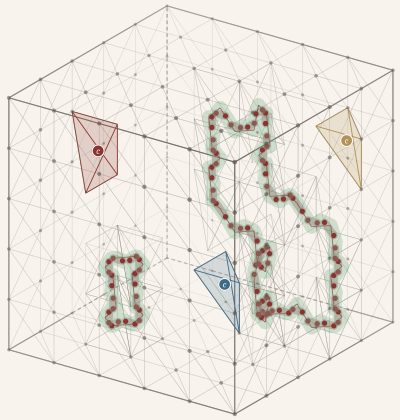
complex fermions c at domain walls (points)

2D



Kitaev chain along the wall

3D



chains on lines, c at points

The same construction works in any dimension: decorate walls and junctions with lower-dimensional invertible states.

Conditions on the decoration

1. **Symmetric**

   The SPT state is invariant under all symmetries.

2. **Uniquely gapped**

   All gapless modes or zero modes at domain defects should vanish.

3. **Retriangulation invariant**

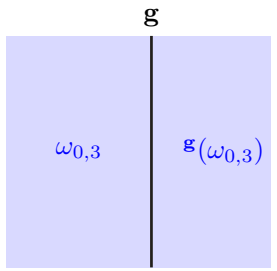
   The constructed fixed-point wavefunction is not affected by different triangulations.

4. **Coherent**

   Different sequences of F moves agree; for example, the (super-)pentagon equation is satisfied.

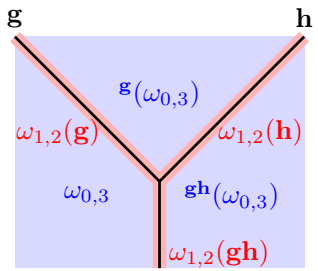
If all four hold, the decorated state is a gapped, symmetric fixed-point wavefunction, i.e. an SPT state.

2D example for bosonic SPT



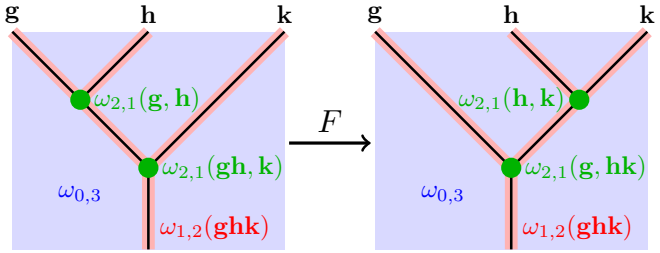
(a) Symmetric

G maps the N-SPT $\omega_{0,3}$ to itself, so the g wall can be gapped.



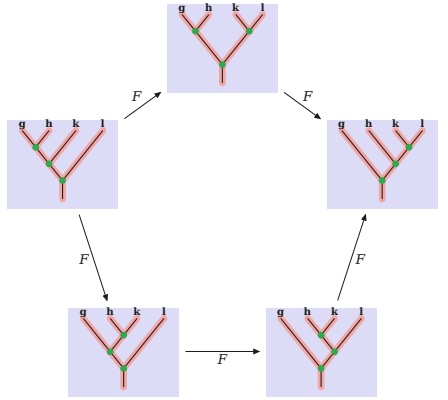
(b) Gapped

The tri-junction carries no projective rep. of N: no zero mode.



(c) Retriangulation invariant

The F move is a symmetric local unitary: it conserves the N charge at the junctions.



(d) Coherent

The two F-move paths of the pentagon agree: no leftover $U(1)$ phase.

Decorations: $\omega_{0,3}$ in the domains, $\omega_{1,2}$ on the walls, $\omega_{2,1}$ at the junctions.

(a)–(c): Fig. 1 of arXiv:2104.13233

Atiyah–Hirzebruch spectral sequence

Concept: The AHSS computes a generalized cohomology h_G^{d+1}, i.e. dD SPT phases (bosonic or fermionic), step by step: start from ordinary group cohomology $E_2^{p,q}$ with lower-dimensional invertible phases as coefficients, remove inconsistent entries page by page with **differentials** d_r, and assemble the survivors E_∞.

$$E_2^{p,q} = H^p\big(Q,\, h^q\big) \overset{d_r}{\Longrightarrow} h_G^{p+q}$$

p	codimension of the Q-defect: domain, wall, point
q	spacetime dimension of the decorating invertible state
d_r	$E_r^{p,q} \longrightarrow E_r^{p+r,\,q-r+1}$ (total degree $+1$)

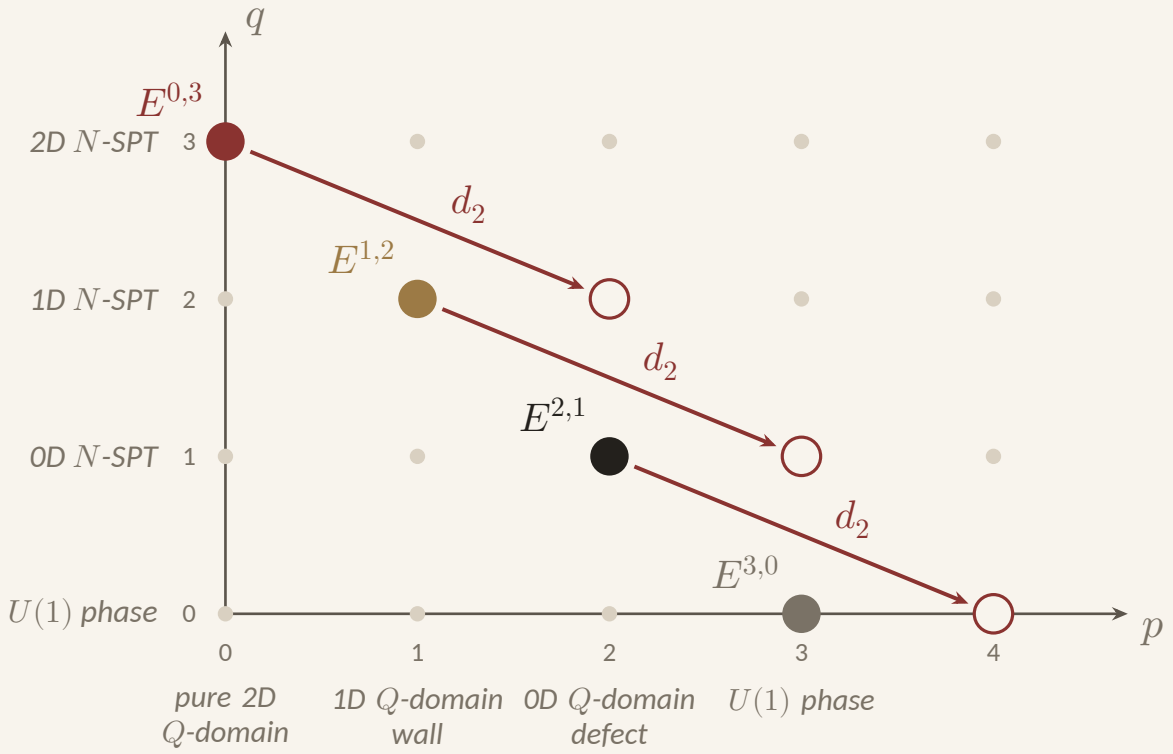
Group-cohomology bosonic SPT $1 \to N \to G \to Q \to 1$

$h^q = H^q(N, U(1))$: N-SPTs as coefficients, N the normal subgroup. This is the **Lyndon–Hochschild–Serre** spectral sequence.

Fermionic SPT $1 \to \mathbb{Z}_2^f \to G_f \to G_b \to 1,\ Q = G_b$

$h^q = h^q(\mathbb{Z}_2^f)$: fermionic invertible phases, i.e., $U(1)$ phase, complex fermion, Kitaev chain, $p+ip$ for $q = 0, 1, 2, 3$.

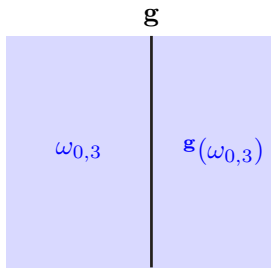
diagonal $p+q=3$: a 2D bosonic SPT, $H^3(G, U(1))$



Each layer $E^{p,3-p}$ sits on the diagonal; d_2 lands on $p+q=4$, where its obstruction (open circle) lives.

Notation: nD $= n$ spatial dimensions

Physical conditions are differentials



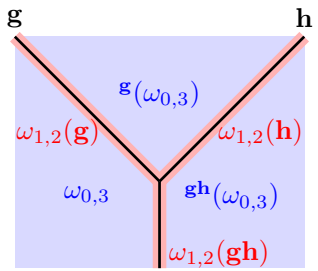
(a) Symmetric

Physical

G maps the N-SPT $\omega_{0,3}$ to itself, so the g wall can be gapped.

Mathematical

$d_1 : E^{0,3} \to E^{1,3}$



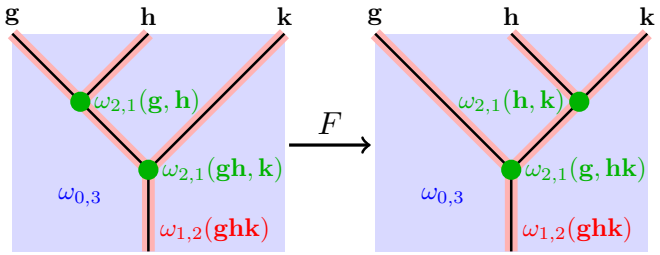
(b) Gapped

Physical

The tri-junction carries no projective rep. of N: no zero mode.

Mathematical

$d_2 : E^{0,3} \to E^{2,2}$



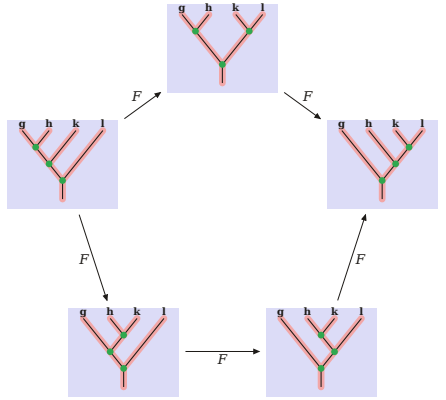
(c) Retriangulation invariant

Physical

The F move is a symmetric local unitary: it conserves the N charge at the junctions.

Mathematical

$d_3 : E^{0,3} \to E^{3,1}$



(d) Coherent

Physical

The two F-move paths of the pentagon agree: no leftover $U(1)$ phase.

Mathematical

$d_4 : E^{0,3} \to E^{4,0}$

Satisfied $\Leftrightarrow$ in $\ker d_r$: the decoration survives to the next page

Violated $\Leftrightarrow$ nonzero image: an obstruction

(a)–(c): Fig. 1 of arXiv:2104.13233; $\omega_{p,q} \in E^{p,q}$ is the decoration on codim-p junctions

The LHS spectral sequence at cochain level

Setting

Consider a group extension $1 \to N \to G \to Q \to 1$.

$$E_2^{p,q} = H^p(Q,\, H^q(N, U(1))) \implies H^{p+q}(G, U(1))$$

1

Group extension

$N \trianglelefteq G$, quotient Q, extension class $\nu \in H^2(Q, N)$

→

2

Cochain-level E_2

Closed cocycle formulas for every $E_2^{p,q}$ entry

→

3

Explicit differentials

Explicit formulas for d_2, d_3, d_4

Q.-R. Wang, **S.-Q. Ning**, M. Cheng, *Domain wall decorations, anomalies and spectral sequences in bosonic topological phases*, arXiv:2104.13233

A differential at cochain level: the 1D case

A 1D SPT of $1 \to N \to \tilde{G} \to G \to 1$, $\nu \in H^2(G,N)$, is a 2-cocycle F of $\tilde{G}$. Cochain ansatz ($a_g = a \cdot 1_g$, N central):

$$F(a_g, b_h) = \underbrace{F_{0,2}(a,b)\, F_{0,2}\big(\nu(g,h), ab\big)}_{E_0^{0,2}}\ \underbrace{F_{1,1}(a;h)}_{E_0^{1,1}}\ \underbrace{F_{2,0}(g,h)}_{E_0^{2,0}}$$

$dF = 1$ splits into four cochain equations

$E_0^{0,3}$ $\quad \delta_0 F_{0,2} = 1$ $\qquad$ $F_{0,2}$ is a cocycle of N

$E_0^{1,2}$ $\quad \delta_1 F_{0,2} \cdot \delta_0 F_{1,1} = 1$ $\qquad$ $\chi_h := F_{1,1}(\cdot\,; h) \in H^1(N, U(1))$

$E_0^{2,1}$ $\quad \underbrace{\dfrac{F_{0,2}(a,\nu(h,k))}{F_{0,2}(\nu(h,k),a)}}_{\delta_2 F_{0,2}} \cdot \underbrace{\dfrac{\chi_{hk}(a)}{\chi_h(a)\chi_k(a)}}_{\delta_1 F_{1,1}} \cdot \delta_0 F_{2,0} = 1$

$E_0^{3,0}$ $\quad \underbrace{\dfrac{F_{0,2}(\nu(g,hk),\nu(h,k))}{F_{0,2}(\nu(gh,k),\nu(g,h))}}_{\delta_3 F_{0,2}} \cdot \underbrace{\chi_k(\nu(g,h))^{-1}}_{\delta_2 F_{1,1}} \cdot \underbrace{d_G F_{2,0}}_{\delta_1 F_{2,0}} = 1$

$F_{0,2} = 1$: $\quad d_2 : E_2^{1,1} \to E_2^{3,0}, \quad d_G F_{2,0}(g,h,k) = \chi_k(\nu(g,h))$

Eqs. (B60)–(B70) of arXiv:2104.13233; MPS: Ning, Ebisu, Lam, arXiv:2603.19381

Take $N = G = \mathbb{Z}_2$, $\chi_h(a) = (-1)^{a\,h}$

$\tilde{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$, $\nu = 0$

$F(a_g, b_h) = (-1)^{a\,h}$

All four lines hold with $F_{0,2} = F_{2,0} = 1$.
$F(x,y)/F(y,x) = -1$: X and Z anticommute on the edge.
The cluster state.

$\tilde{G} = \mathbb{Z}_4$, $\nu(g,h) = gh$ (carry)

$d_G F_{2,0} = (-1)^{g\,h\,k}$

The generator of $H^3(\mathbb{Z}_2, U(1))$, never a coboundary: the $E_0^{3,0}$ line has no solution. Obstructed, as $H^2(\mathbb{Z}_4, U(1)) = 0$.

Other dimensions: a $(d{+}1)$-cocycle splits into $d{+}2$ layers.

Why the cochain-level LHS spectral sequence?

Group-cohomology bosonic SPTs are already classified by $H^{p+q}(G, U(1))$. Why do we still need the cochain-level LHS spectral sequence?

1 Finer structure of the SPT wavefunction

Each class in $H^{p+q}(G, U(1))$ is resolved into layers $E^{p,q}$: which lower-dimensional N-SPT sits on which Q-defect.

The layer structure is also useful for **disordered** and **decoherent** (mixed-state) SPT phases, where only part of the symmetry is exact.

R. Ma, C. Wang, PRX **13**, 031016 (2023)
R. Ma, J.-H. Zhang, Z. Bi, M. Cheng, C. Wang, PRX **15**, 021062 (2025)
J. Y. Lee, Y.-Z. You, C. Xu, Quantum **9**, 1607 (2025)

2 Applications beyond SPT

LSM constraints: take Q to be translations; the unit-cell content is a class on E_2, and the differentials decide LSM vs. SPT-LSM.

SET phases: gauging N turns the SPT into an SET with gauge group N, enriched by the Q symmetry. The decoration data become symmetry fractionalization, and obstructions become quantum anomalies.

Next: from bosonic to fermionic SPT

Fermionic SPT: focus on 3D

The same spectral-sequence logic, now with fermionic layers: interacting crystalline topological superconductors in three dimensions, for all 230 space groups together with internal symmetries.

Remark. Due to the computational cost and other potential analytic complexity, we want the obstruction conditions at **cochain level**: they make large groups computable, such as all 230 space groups and the magnetic groups.

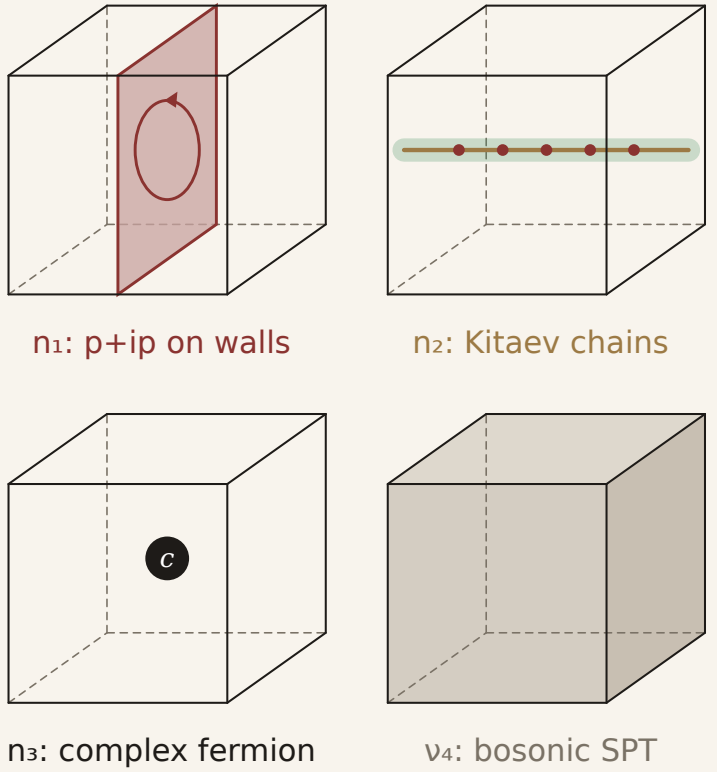
S.-Q. Ning, X.-Y. Ren, Q.-R. Wang, Y. Qi, Z.-C. Gu, arXiv:2512.25069

Four layers of decoration data

Via the **crystalline equivalence principle** it suffices to classify fermionic SPTs with internal symmetry $G_b \times_{\omega_2} \mathbb{Z}_2^f$. The domain-wall construction extends to fermions: a 3D FSPT is characterized by four layers of data.

DATA	DECORATED STRATUM	DECORATION
n_1	Domain walls (codim-1)	$p + ip$ superconductor
n_2	Codim-2 junctions	Majorana (Kitaev) chain
n_3	Codim-3 junctions	Complex fermion
ν_4	Bulk (3+1)D	Bosonic SPT layer

n_1 needs special care (see the $p + ip$ slide). For (n_2, n_3, ν_4) the obstruction formulas are explicit, so they can be evaluated by computer.



Three layers that can be computed automatically

Without the $p + ip$ layer ($n_1 = 0$), the remaining three layers (n_2, n_3, ν_4) on the diagonal $p + q = 4$ have explicit obstructions and can be computed automatically.

$$dn_2 = 0$$

uniquely gapped: Kitaev-chain ends pair up

$$dn_3 = O_4[n_2]$$

retriangulation invariance

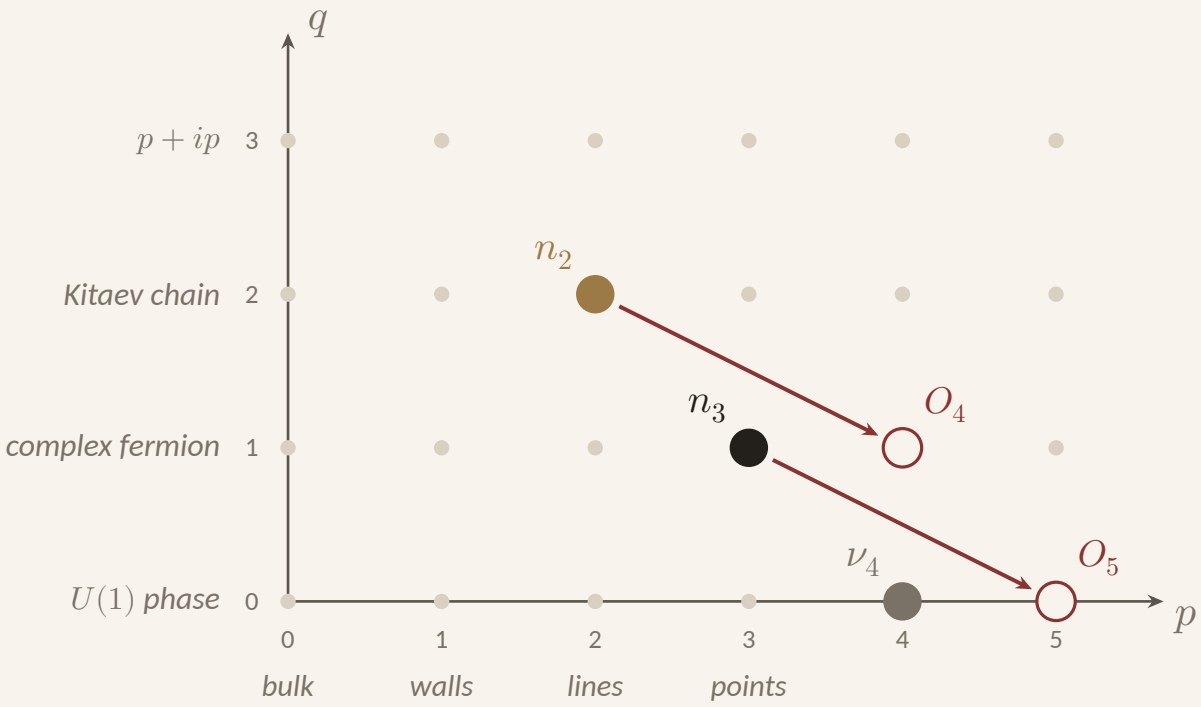
$$d\nu_4 = O_5[n_2, n_3]$$

coherence: super-pentagon

Each d_2 lands on the next diagonal, where the obstruction lives.

Notation: nD $= n$ spatial dimensions

diagonal $p + q = 4$: a 3D fermionic SPT



Source: arXiv:2512.25069

Additional treatment of the $p + ip$ layer: two sources

Source 1 · **Anti-unitary symmetries**

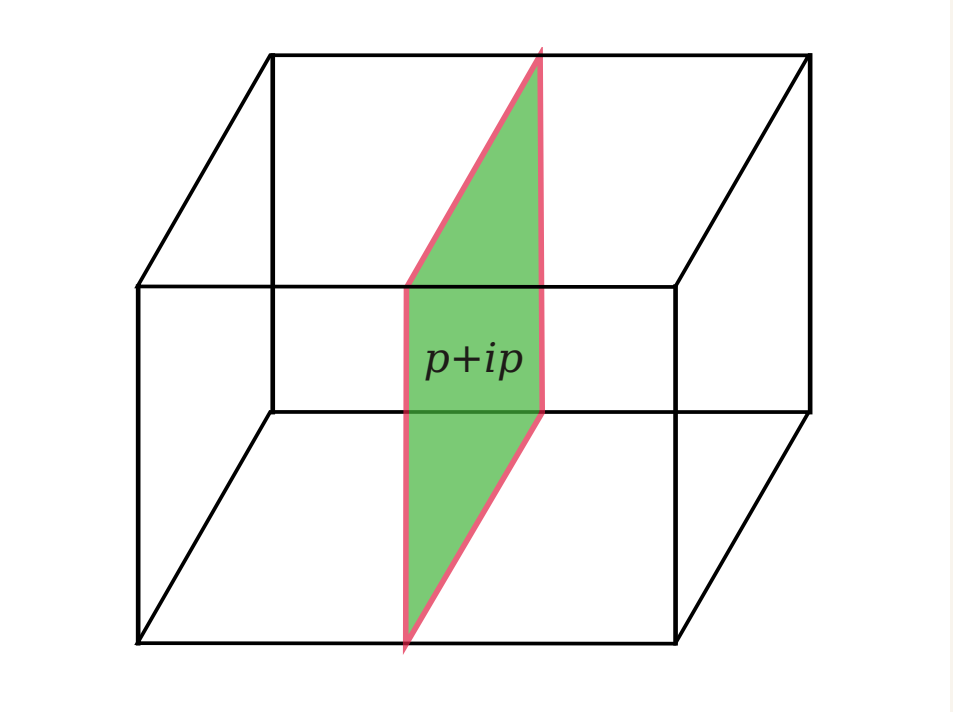
Time-reversal-like walls $\longleftrightarrow$ mirror planes with $p + ip$

For anti-unitary symmetries the n_1 classification maps to a **mirror FSPT** problem: a mirror plane carrying a $p + ip$ superconductor.

Source 2 · **Infinite-order symmetries**

Translations and weak TSCs

For translation-like symmetries the $p + ip$ layer is fixed by the free part of $H^1(G_b, \mathbb{Z}_T)$; these data are precisely the **weak topological superconductors**.



A $p + ip$ superconductor on the mirror plane: it is compatible with the mirror symmetry G_f^m if $\omega_2^m = \omega_2 + s_1 \cup s_1$ is a coboundary.

Figure adapted from Fig. 2 of arXiv:2512.25069

Classification for all 230 space groups

Crystalline equivalence

The **crystalline equivalence principle** trades spatial symmetries for internal ones: the internal-symmetry FSPT classification yields interacting topological crystalline superconductors for **all 230 space groups**.

No.	Int.	$E_{2\mathrm{D}}$ $p{+}ip$	$E_{1\mathrm{D}}$ Kitaev	$E_{0\mathrm{D}}$ fermion	E_{b} bosonic	Total
1	$P1$	$\mathbb{Z}^3$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$16 \times \mathbb{Z}^3$
2	$P\bar{1}$	$\mathbb{Z}^3$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2$	$32 \times \mathbb{Z}^3$
3	$P2$	$\mathbb{Z}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$2048 \times \mathbb{Z}$
10	$P2/m$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^6$	$2048 \times \mathbb{Z}$
62	$Pnma$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	4
65	$Cmmm$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{13}$	16384
122	$I\bar{4}2d$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	256
123	$P4/mmm$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^{18}$	262144

$E_{2\mathrm{D}}, E_{1\mathrm{D}}, E_{0\mathrm{D}}, E_{\mathrm{b}}$: $p{+}ip$, Kitaev-chain, complex-fermion and bosonic-SPT layers; $\mathbb{Z}_1$ = trivial; Total = full interacting classification.

Selected rows, Tables 1–2 of arXiv:2512.25069

Summary

1

The decorated domain wall construction is organized by the Atiyah–Hirzebruch spectral sequence; each term and differential has a physical meaning.

2

We gave cochain-level formulas for the LHS spectral sequence and used them for anomalies, LSM theorems and SET classifications.

3

For fermions, four layers of decoration data give the classification of interacting crystalline TSCs for all 230 space groups.

Thank you!

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