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QMATH

A non-robust quantum correlation self-test

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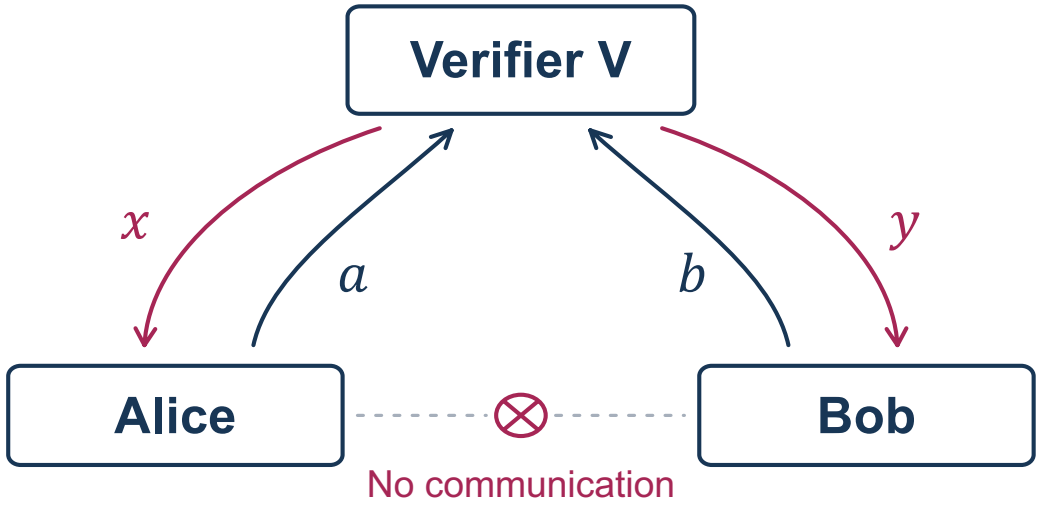
arXiv:2609.25117

Bell experiments and self-testing

The Bell scenario

The verifier sends a question to each device. Alice and Bob each return one answer.



- A strategy specifies a shared state ψ and local measurement operators $M_{x,a}, N_{y,b}$
- The Born rule predicts the probability of each pair of answers:

$$p_S(a, b \mid x, y) = \langle \psi \mid M_{x,a} \otimes N_{y,b} \mid \psi \rangle$$

Self-testing

For some correlations, there is a ‘unique’ ideal state and measurement actions. [1]

[1] Ivan Šupić, Joseph Bowles, “Self-testing of quantum systems: a review”

The CHSH game and its target correlation

Game rule

- Alice and Bob receive independent, uniformly random **bits** x, y and return **bits** a, b.
- They win when $a \oplus b = x \wedge y$.

The maximum classical win probability of CHSH is 75%.

The following correlation p_* wins with probability $\cos^2(\pi/8) \approx 85.4\%$ on every input pair. [3]

Inputs	Output 00	Output 01	Output 10	Output 11	Win rate
00, 01, or 10	**½ cos²(π/8)**	½ sin²(π/8)	½ sin²(π/8)	**½ cos²(π/8)**	cos²(π/8)
11	½ sin²(π/8)	**½ cos²(π/8)**	**½ cos²(π/8)**	½ sin²(π/8)	cos²(π/8)

[3] Y. Wang, X. Wu, and V. Scarani, “All the self-testings of the singlet for two binary measurements,” *New J. Phys.* **18**, 025021 (2016).

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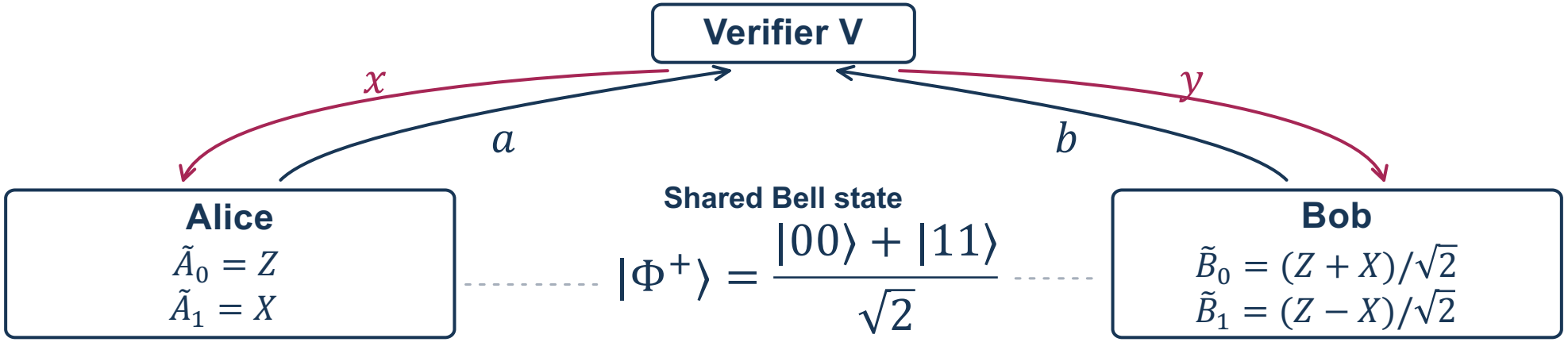
11	½ sin²(π/8)	**½ cos²(π/8)**	**½ cos²(π/8)**	½ sin²(π/8)	cos²(π/8)

There is a ‘unique’ strategy that implements p_* .

[3] Y. Wang, X. Wu, and V. Scarani, “All the self-testings of the singlet for two binary measurements,” *New J. Phys.* **18**, 025021 (2016).

The strategy certified by the CHSH correlation

This strategy realizes the target table p_* [3]



Each input selects an observable. Outcomes +1 and −1 give answer bits 0 and 1.

[3] Y. Wang, X. Wu, and V. Scarani, “All the self-testings of the singlet for two binary measurements,” *New J. Phys.* **18**, 025021 (2016).

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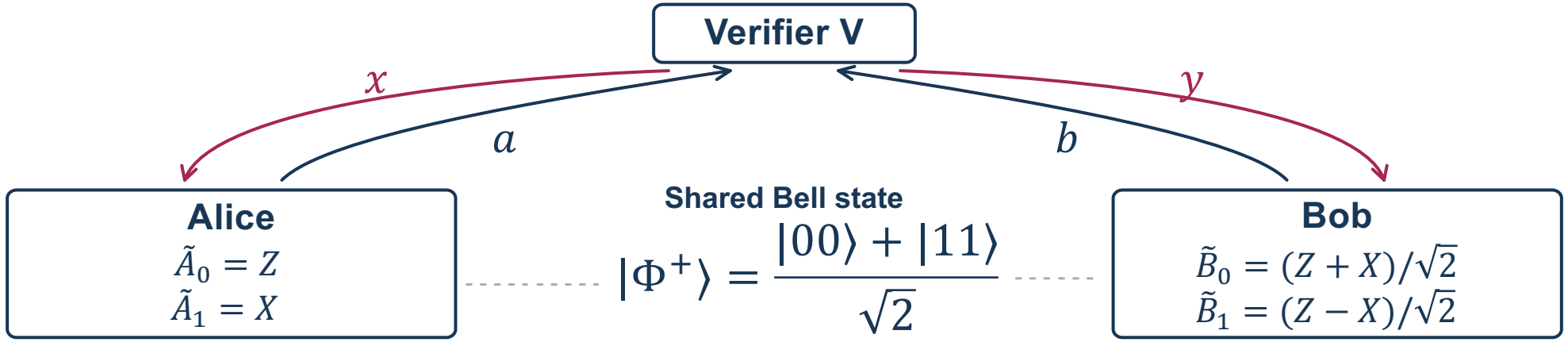

Each input selects an observable. Outcomes +1 and −1 give answer bits 0 and 1.

Self-testing (usual definition). Every realization of p_* gives this Bell pair and Pauli measurements, up to local isometries and an auxiliary state: [3]

$$(V_A \otimes V_B)|\psi\rangle = |\Phi^+\rangle \otimes |\text{aux}\rangle,$$
$$V_A A_0 V_A^* = Z \otimes I,$$
$$V_A A_1 V_A^* = X \otimes I. \text{(similar for Bob)}$$

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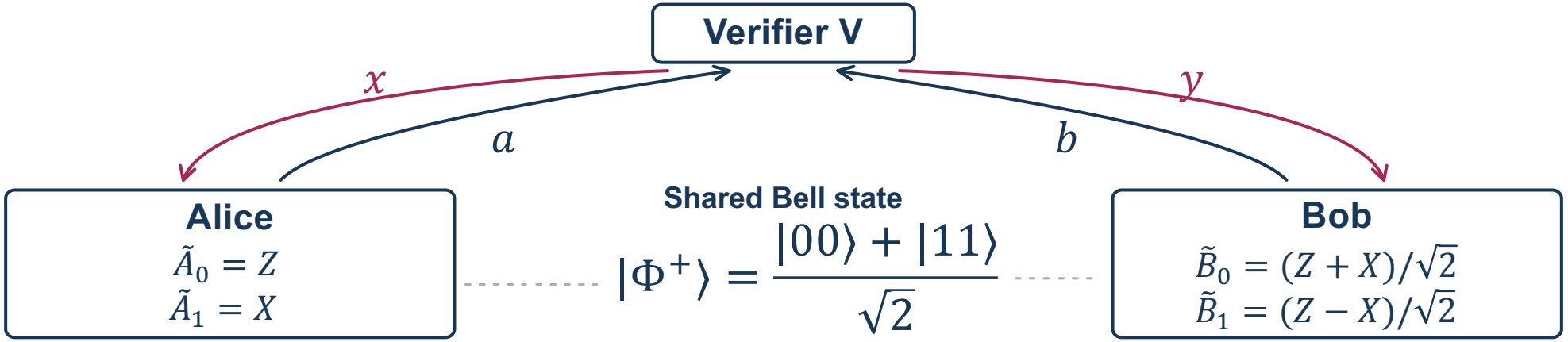

Each input selects an observable. Outcomes +1 and −1 give answer bits 0 and 1.

Self-testing (robust version). Every close realization $p \approx_\epsilon p_*$ gives approximately this Bell pair and Pauli measurements, up to local isometries and an auxiliary state:

$$(V_A \otimes V_B)|\psi\rangle \approx_{f(\epsilon)} |\Phi^+\rangle \otimes |\text{aux}\rangle,$$
$$V_A A_0 V_A^* \approx_{f(\epsilon)} Z \otimes I,$$
$$V_A A_1 V_A^* \approx_{f(\epsilon)} X \otimes I. \text{(similar for Bob)}$$

The moment table

An operator-algebraic viewpoint of self-testing

The observed corner of the moment table

Rows are indexed by words u in Alice's operators. Columns are indexed by words v in Bob's. [2]

$$m_S(u,v) = \langle \psi \mid u \otimes v \mid \psi \rangle.$$

Alice's words ↓	Bob's words →			
	I	B_j	B_jB_l	…
I	1	$p_B(j)$	?	…
A_i	$p_A(i)$	$p_{ij} = \langle\psi\|A_i \otimes B_j\|\psi\rangle$	?	…
A_iA_k	?	?	?	…
⋮	⋮	⋮	⋮	⋱

Blue block · $i = (x,a), j = (y,b), p_{ij} = p(a,b \mid x,y)$.
A higher moment is, e.g., $m_{ik,j} = \langle A_iA_k \otimes B_j \rangle$.

- The blue block gives joint probabilities and marginals, which can be **obtained from experiments.**
- A product such as A_iA_k can involve two or more inputs for Alice. It is **unobserved**.

[2] C. Paddock et al., "An operator-algebraic formulation of self-testing," *Ann. Henri Poincaré* **25**, 4283–4319 (2024).

Exact self-testing fixes the completion

Self-testing: the up-left corner fix the whole moment table.

Alice's words ↓	Bob's words →			
	I	B_j	B_jB_l	…
I	1	$p_B(j)$	$m^\star_{0,jl}$	…
A_i	$p_A(i)$	p_{ij}	$m^\star_{i,jl}$	…
A_iA_k	$m^\star_{ik,0}$	$m^\star_{ik,j}$	$m^\star_{ik,jl}$	…
⋮	⋮	⋮	⋮	⋱

$m^\star_{ik,j}$ is the ideal value of $\langle A_iA_k \otimes B_j \rangle$.

Self-testing

The implementing strategies of the correlation $p(a,b|x,y)$ agree on all higher-order moments.

For those familiar with operator algebra:

Equivalently, exactly one finite-dimensional state on $\mathcal{U}_A \otimes_{\min} \mathcal{U}_B$ implements p. [2] Here

$$\mathcal{U}_A = C^*_{\text{univ}} \langle e_{x,a} \mid e_{x,a} \geq 0, \sum_a e_{x,a} = 1 \text{ for each } x \rangle.$$

[2] C. Paddock et al., "An operator-algebraic formulation of self-testing," *Ann. Henri Poincaré* **25**, 4283–4319 (2024).

Robustness stabilizes each fixed moment

Let S_n be finite-dimensional strategies with observed tables $p_{S_n} \to p$, where p is a robust self-test. [5]

Alice's words ↓	Bob's words →			
	I	B_j	B_jB_l	$\cdots$
I	1	$p_B^{(n)}(j) \to p_B(j)$	$m_{0,jl}^{(n)} \to m_{0,jl}^{\star}$	$\cdots$
A_i	$p_A^{(n)}(i) \to p_A(i)$	$p_{ij}^{(n)} \to p_{ij}$	$m_{i,jl}^{(n)} \to m_{i,jl}^{\star}$	$\cdots$
A_iA_k	$m_{ik,0}^{(n)} \to m_{ik,0}^{\star}$	$m_{ik,j}^{(n)} \to m_{ik,j}^{\star}$	$m_{ik,jl}^{(n)} \to m_{ik,jl}^{\star}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Fix a block (u, v)

- As the observed errors vanish, $m_{S_n}(u, v) \to m_*(u, v)$.
- The required accuracy may depend on the words u and v.

For those familiar with operator algebra:

Equivalently, exactly one state on $\mathcal{U}_A \otimes_{\min} \mathcal{U}_B$ implements p, not necessarily finite dimensional. [5,6]

[5] Y. Zhao, "Robust self-testing for nonlocal games with robust game algebras," arXiv:2411.03259 (2024).
[6] P. N. Kar, "Robust self-testing for synchronous correlations and games," arXiv:2503.23500 (2025).

The robustness question

Does exact self-testing of a single quantum correlation imply robust self-testing?

Robustness in self-testing

- There was no non-robust self-test (before this work).
- For a wide range of games, self-testing is automatically robust. [5]

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From operator-algebraic viewpoint:

Self-test = Rigidity at p

- Every finite-dimensional strategy giving the exact table has the same moment completion.

Robustness = Stability near p

- Along every sequence $p_{S_n} \to p$, each fixed moment approaches its ideal value.

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- Along every sequence $p_{S_n} \to p$, each fixed moment approaches its ideal value.

> **To find a counterexample:**
>
> Construct universal C* algebra with unique fin. dim. state, and another inf. dim. state.

[5] Y. Zhao, “Robust self-testing for nonlocal games with robust game algebras,” arXiv:2411.03259 (2024).

Main result

There is a synchronous binary quantum correlation that self-tests its ideal strategy exactly and fails to do so robustly.

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

Synchrony and maximally entangled blocks

Synchronous correlation (definition)

Alice and Bob share the input and output sets. Equal inputs give equal outputs.

$p(a, b|x, x) = 0$ for every x and $a \neq b$.

In a finite-dimensional synchronous strategy [7]

- Without loss of generality, one can assume that the players use direct sum of maximally entangled blocks.
- And both players use same measurement operators (up to transpose).

The joint probabilities become normalized tracial traces

$$p(a, b \mid x, y) = \langle \Phi_d \mid P_{x,a} \otimes P_{y,b}^{\mathsf{T}} \mid \Phi_d \rangle = \frac{1}{d} \mathrm{Tr}(P_{x,a} P_{y,b}).$$

[7] V. I. Paulsen et al., “Estimating quantum chromatic numbers,” *J. Funct. Anal.* **270**, 2188–2222 (2016).

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For a synchronous quantum correlation

Consider implementing tratial states τ on a **single** universal C* algebra:

$$\mathcal{U}_{\mathrm{PVM}} = C^*_{\mathrm{univ}}\langle e_{x,a} \mid e_{x,a} = e^*_{x,a} = e^2_{x,a}, e_{x,0} + e_{x,1} = 1(x \in X, a = 0,1)\rangle, \qquad p(a,b|x,y) = \tau(e_{x,a}e_{y,b}).$$

Exact self-test	$\Leftrightarrow$	A unique **finite-dimensional** tracial states implements p
Robust self-test	$\Leftrightarrow$	A unique **amenable** tracial states implements p

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Our goal:

Find a correlation p and a universal C* algebra such that

- p has a unique finite dimensional implementing tratial state
- p has a different (infinite-dimensional) amenable implementing tratial state

[7] V. I. Paulsen et al., “Estimating quantum chromatic numbers,” *J. Funct. Anal.* **270**, 2188–2222 (2016).

Step 1. Encode a finitely-presented algebra

Let $\mathcal{A}$ be a unital C^* -algebra with a tracial state and a finite real-coefficient unitary presentation.

Theorem 4.2 (Encoding by projections) [1]

There are finitely many projections $P_x \in M_N(\mathcal{A})$ with three properties.

- **(a) Generation.** $\Phi: e_{x,1} \mapsto P_x$ maps $\mathcal{U}_{\mathrm{PVM}}$ onto $M_N(\mathcal{A})$.
  - So, distinct tracial states of $\mathcal{A}$ stay distinct in $\mathcal{U}_{\mathrm{PVM}}$ under pullback.
- **(b) Common data.** Every tracial state τ on $\mathcal{A}$ gives the same values $(\mathrm{tr}_N \otimes \tau)(P_x P_y)$
  - So, the correlation (up-left corner) $p_{\mathcal{A}}$ is determined.
- **(c) Recovery.** Every tracial state on $\mathcal{U}_{\mathrm{PVM}}$ implementing the correlation has the form $(\mathrm{tr}_N \otimes \tau) \circ \Phi$ for a unique trace τ on $\mathcal{A}$.
  - So, uniqueness is preserved.

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

Step 1. Encode a finitely-presented algebra

Our goal becomes:

Find a correlation p and a universal C* algebra such that

- p has a unique finite dimensional implementing tratial state
- p has a different (infinite-dimensional) amenable implementing tratial state



Find a finitely presented C* algebra such that

- It has a unique finite dimensional tratial state
- It has a different (infinite-dimensional) amenable tratial state

Step 2. An algebra with two kinds of traces

Proposition 6.1 (Irrational rotation construction) [1]

Let $\mathcal{A}_{\mathrm{rot}}$ be the unital C^*-algebra on generators u, v, z with

$$zu = uz, zv = vz, uv = zvu,$$
$$(2z + 2z^* - I)(u + v - 2I) = 0.$$

The character $\tau_0 : u, v, z \mapsto 1$ is its unique finite-dimensional trace. There is a distinct amenable trace τ_1.

- **Scalar component.** In an irreducible representation, central z is a scalar λ. If $2\lambda + 2\bar{\lambda} - 1 \neq 0$, the last relation forces $u = v = 1$ and then $\lambda = 1$.
- **Rotation components.** Otherwise $\lambda = \zeta$ or $\bar{\zeta}$, where $\zeta = (1 + i\sqrt{15})/4$. What remains is $uv = \lambda vu$.

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

Further consequences and an open problem

The encoding also gives an infinite-dimensional self-test and an extreme correlation with many inequivalent realizations. [1, §7]

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

An infinite-dimensional self-test

The modified algebra $B \cong M_2(A_\zeta)$ has a unique trace, which is amenable. It has no finite-dimensional representation.

- Finite-dimensional strategies approximate p_B arbitrarily closely.
- No spatial tensor-product strategy attains the limit, even in infinite dimensions. The finite-dimensional set is therefore not closed.
- In the commuting-operator model, Alice's and Bob's operators commute on one Hilbert space. Exactly one state implements p_B.

> The limit p_B is a commuting-operator self-test and is extreme in C_{qc}. [1, Prop. 7.1]

[1] R. Chen and Y. Zhao, "A non-robust quantum correlation self-test," arXiv:2609.25117 (2026).

One extreme correlation, many realizations

An extreme correlation admits no nontrivial mixture of distinct correlations.

- **Two realizations.** The two characters of $\mathbb{C} \bigoplus \mathbb{C}$ give one synchronous binary table, extreme in C_{qc}. Neither strategy is a local dilation of the other.
- **A continuum.** For $C(\mathbb{T})$, points on the unit circle give pairwise inequivalent realizations of one extreme table.

> All these strategies share one maximally entangled state and one finite local dimension N. [1, §7.2]

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

Open problem: the smallest counterexample

For extreme synchronous binary correlations, our construction is still well above the lower bounds. [1, Cor. 6.2; Prop. 6.3]

	Questions per party	Schmidt rank of the ideal state
Proved lower bound	4	3
Our construction	250	36 (can be improved to 18)

> **Open problem.** Determine the smallest possible number of questions and the smallest possible ideal Schmidt rank for this separation.

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

Summary

> **Exact self-testing of a single quantum correlation does not imply robustness.** [1, Thm. 1.3]

- In our synchronous binary example, the observed probabilities determine a unique finite-dimensional moment table.
- An irrational rotation yields a second amenable trace. It agrees on the observed corner and differs on higher moments.

Two consequences of the encoding [1, §7]

- An infinite-dimensional self-test in the commuting-operator model.
- One extreme correlation with a continuum of inequivalent finite-dimensional realizations.

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

Thank you

A non-robust quantum correlation self-test

Ranyiliu Chen and Yuming Zhao

arXiv:2609.25117

Operator-algebraic characterizations

General POVMs · extreme p, finite-dimensional pure-state strategies [1, Thms. 3.4–3.5]

- A state on $\mathcal{U}_A^{\mathrm{POVM}} \otimes_{\min} \mathcal{U}_B^{\mathrm{POVM}}$ implements p if $\omega(e_{x,a} \otimes f_{y,b}) = p(a,b \mid x,y)$
- Exact self-testing is equivalent to a unique finite-dimensional implementing state.
- Robust self-testing implies uniqueness among all implementing states. The general converse is open.

Extreme synchronous correlations · projective measurements [1, Thm. 3.8]

- On the minimal tensor product of the PVM algebras, both equivalences hold.
- Implementing states correspond to amenable traces on one PVM algebra. This correspondence preserves finite-dimensionality.

Under the encoding, Connes-embeddable traces on the input algebra correspond to amenable implementing traces.

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

The four rotation relations

Relations for the unitaries u, v, z [1, Prop. 6.1]

$$zu = uz, zv = vz, uv = zvu,$$
$$(2z + 2z^* - I)(u + v - 2I) = 0.$$

- On a d-dimensional μ-eigenspace of z, determinants give $\mu^d = 1$. The algebraic integer $\mu + \mu^{-1}$ cannot equal $1/2$.
- The first factor is nonzero, so $u + v = 2I$. Unitarity forces $u = v = I$, and hence $\mu = 1$.
- At $\zeta = (1 + i\sqrt{15})/4$ the first factor vanishes. The irrational rotation quotient $uv = \zeta vu$ supplies the other trace.

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

The traces implementing the encoded table

Theorem 5.1 (Trace correspondence) [1]

The synchronous binary table $p_{\mathcal{A}}$ is extreme in C_{qc}. The pullback

$$\Theta(\tau) = (\mathrm{tr}_N \otimes \tau) \circ \Phi$$

is an affine weak-* homeomorphism from traces on $\mathcal{A}$ to **all** traces implementing $p_{\mathcal{A}}$.

Trace on the input algebra	Implementing trace of the table
Finite-dimensional	Finite-dimensional
Connes-embeddable	Amenable

Goal. An algebra with a unique finite-dimensional trace and a second Connes-embeddable trace.

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

The rotation component gives the second trace

The three central values give $\mathcal{A}_{\mathrm{rot}} \cong \mathbb{C} \oplus A_\zeta \oplus A_{\bar{\zeta}}$ In each rotation algebra A_λ , the phase $\lambda \in \{\zeta, \bar{\zeta}\}$ is not a root of unity. [1, §6.1]

Finite-dimensional representations

$$UV = \lambda VU \Longrightarrow \det(UV) = \lambda^r \det(VU) \Longrightarrow \lambda^r = 1.$$

Neither rotation component can occur in dimension r. Every finite-dimensional trace therefore comes from the scalar component.

An amenable trace

The irrational rotation algebra A_ζ is nuclear, so its canonical trace is amenable. Pulling it back gives $\tau_1(z) = \zeta$, while $\tau_0(z) = 1$.

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

Two traces, one table, and the separation

Apply the encoding to $\mathcal{A}_{\mathbf{rot}}$ The traces $\Theta(\tau_0)$ and $\Theta(\tau_1)$ agree on every entry of the extreme table $p_{\mathbf{rot}}$ [1, Cor. 6.2]

> **They differ on an unobserved element.** Since Φ is onto, choose h with $\Phi(h) = I_N \otimes z$. Then
>
> $$\Theta(\tau_0)(h) = 1, \Theta(\tau_1)(h) = \zeta.$$

- **Exact.** By recovery and the finite-dimensional correspondence, $\Theta(\tau_0)$ is the only finite-dimensional implementing trace.
- **Non-robust.** τ_1 is amenable, hence Connes-embeddable. Its image is a second amenable implementing trace.

> **Ideal strategy.** Evaluating at the character gives projections in $M_{36}(\mathbb{C})$ and the maximally entangled state $|\Phi_{36}\rangle$Each player has 250 binary measurements.

Schmidt blocks in synchronous strategies

For a finite-dimensional pure-state realization, restrict to its Schmidt supports. Synchrony makes the measurements projective there. [7]

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}\,|ii\rangle, D = \mathrm{diag}(\sqrt{\lambda_i}) > 0.$$

Perfect agreement identifies the measurement actions

$$(P_{x,a} \otimes I)\,|\psi\rangle = (I \otimes Q_{x,a})\,|\psi\rangle \Longrightarrow P_{x,a}D = DQ_{x,a}^{\mathsf{T}}.$$

- Taking adjoints gives $DP_{x,a} = Q_{x,a}^{\mathsf{T}}D$, hence $[P_{x,a}, D^2] = 0$.
- Both measurements preserve the equal-eigenvalue blocks of D. On block k, $D = c_k I_{d_k}$, so $Q_{x,a}^{(k)} = (P_{x,a}^{(k)})^{\mathsf{T}}$.

$$|\psi\rangle = \oplus_k \sqrt{w_k}\,|\Phi_{d_k}\rangle, w_k = d_k c_k^2, p = \sum_k w_k\, p_k.$$

If p is extreme, every p_k with $w_k > 0$ equals p.

[7] V. I. Paulsen et al., “Estimating quantum chromatic numbers,” *J. Funct. Anal.* **270**, 2188–2222 (2016).

Inside the recovery argument

The visible pair moments determine $\varphi(L^*L)$ for every linear combination $L = c_0 I + \sum_x c_x P_x$. In the tracial GNS representation, zero norm forces $L = 0$. [1, §4]

- **Recover the generators.** Two-coordinate projection blocks put each unitary in an off-diagonal entry. The auxiliary projection identities recover matrix units and a common corner containing the unitaries.
- **Identify repeated letters.** An auxiliary projection T compares two occurrences U, H of the same generator. The equality $(T^2 - T)_{14} = (U - H)/8 = 0$ forces them to agree.
- **Read each relation.** Recursion fixes the column of words. Its overlap with a fixed coefficient column tests $R_j(U) = 0$. The universal property now gives the required factorization.

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

The commutation example in coordinates

For two unitaries u, v, the word column and coefficient column in Example 4.7 are [1, Ex. 4.7]

$$\eta = \frac{1}{\sqrt{5}}(0,1,v,uv,u,vu)^{\mathsf{T}}, \xi = \frac{1}{\sqrt{3}}(1,0,0,1,0,-1)^{\mathsf{T}}.$$

Both columns are isometries. Their range projections satisfy

$$E = \eta\eta^*, F = \xi\xi^*, EF = 0 \iff \xi^*\eta = 0 \iff uv = vu.$$

- The diagonal overlaps are both $1/6$ under $\mathrm{tr}_6 \otimes \tau$.
- The off-diagonal overlap vanishes for commuting u, v. For the anticommuting Pauli pair X, Z , it equals $2/45$ under $\mathrm{tr}_6 \otimes \mathrm{tr}_2$

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

Why the encoded table is extreme

Let $J = \ker \Phi$. Theorem 4.2 identifies the traces implementing $p_{\mathcal{A}}$ with exactly the traces that factor through the quotient by J . [1, proof of Thm. 5.1]

- Suppose $p_{\mathcal{A}} = tp_1 + (1-t)p_2$ in C_{qc}, with $0 < t < 1$. Zero disagreement on equal inputs forces both tables to be synchronous. Choose implementing traces φ_1, φ_2.
- Their mixture factors through Φ. For $c \in J$, its value on c^*c is zero. Positivity forces $\varphi_1(c^*c) = \varphi_2(c^*c) = 0$
- Both traces therefore factor through Φ. Part (b) gives $p_1 = p_2 = p_{\mathcal{A}}$

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).

Amenability and finite matrix approximations

A trace τ on a separable unital C^*-algebra $\mathcal{A}$ is amenable exactly when unital completely positive maps $\psi_k: \mathcal{A} \to M_{n_k}$ satisfy, for all $a, b \in \mathcal{A}$, [8, Thm. 3.1.6]

$$\mathrm{tr}_{n_k}(\psi_k(a)) \to \tau(a), \| \psi_k(ab) - \psi_k(a)\psi_k(b) \|_{2,\mathrm{tr}_{n_k}} \to 0.$$

The rotation relation in finite matrices

Choose roots of unity $\lambda_k \to \zeta$. The clock and cyclic shift matrices satisfy $U_k V_k = \lambda_k V_k U_k$. Hence

$$\| U_k V_k - \zeta V_k U_k \| = | \lambda_k - \zeta | \longrightarrow 0.$$

This illustrates approximation of the relation. Nuclearity of A_ζ gives amenability of its trace. [1, §6.1]

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).
[8] N. P. Brown, “Invariant means and finite representation theory of C*-algebras,” *Mem. Amer. Math. Soc.* **184**, no. 865 (2006).

Why the ideal Schmidt rank is 36

The encoding uses $d = 3$ unitary generators, 13 distinct monomials in the relations, and 14 words after closing under suffixes. [1, Eq. (4.17) and Cor. 6.2]

$$N = (2d + 3) + |W| + |S| = 9 + 14 + 13 = 36.$$

- Evaluating the entries of each encoded projection at τ_0 maps $M_N(\mathcal{A}_{\mathrm{rot}})$ onto $M_N(\mathbb{C})$. Write $B_{x,a}$ for the resulting projection.
- Alice uses $B_{x,a}$ and Bob uses $B_{y,b}^{\mathsf{T}}$. The normalized matrix trace is realized by the maximally entangled state

$$|\Phi_{36}\rangle = \frac{1}{6}\sum_{i=1}^{36} |ii\rangle, p_{\mathrm{rot}}(a, b \mid x, y) = \mathrm{tr}_{36}(B_{x,a}B_{y,b}).$$

[1] R. Chen and Y. Zhao, “A non-robust quantum correlation self-test,” arXiv:2609.25117 (2026).