An elliptic deformation of continued fractions

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November 23, 2026

Shenzhen–Nagoya Workshop on Quantum Science 2026

Rational numbers, continued fractions, and their q-analogue

For $x \in \mathbb{Q}_{>1}$, the ordinary regular and negative continued fractions are

$$x \;=\; [a_1, a_2, \ldots] \coloneqq a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}} \;=\; [\![c_1, c_2, \ldots]\!] \coloneqq c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \cdots}},$$

with $a_i \geq 1$ and $c_i \geq 2$. For instance,

$$\frac{5}{2} = 2 + \frac{1}{2} = [2,2] = 3 - \frac{1}{2} = [\![3,2]\!].$$

q-rationals and q-continued fractions of Morier-Genoud and Ovsienko[1] replace these evaluations by q-deformed ones, whose limit under $q \to 1$ recover the ordinary ones.

$$[5/2]_q \coloneqq \frac{1+2q+q^2+q^3}{1+q} = \underbrace{1 + q + \frac{q^2}{1+q^{-1}}}_{\text{regular}} = \underbrace{1 + q + q^2 - \frac{q^2}{1+q}}_{\text{negative}}.$$

[1] S. Morier-Genoud and V. Ovsienko, *q-deformed rationals and q-continued fractions*, Forum Math. Sigma **8** (2020), e13, 55 pp.

Morier-Genoud–Ovsienko q-rationals

Morier-Genoud–Ovsienko q-ratinonals. For every $x \in \mathbb{Q}_{>0}$, a rational function in q:

$$[x]_q = \frac{n_x(q)}{d_x(q)}, \quad \text{such that, for } m \in \mathbb{Z}_{>0},\ [m]_q = 1 + q + \cdots + q^{m-1},\ q\text{-integer}.$$

- For $x > 1$, the regular and negative q-continued fractions give the same $[x]_q$.
- $n_x(q), d_x(q)$ have descriptions using the Stern–Brocot tree and type-A quivers.
- $[x]_q$ enjoys nice properties: Farey-sum recursions, inversion formula, positivity, ...

Relation to several branches of mathematics:

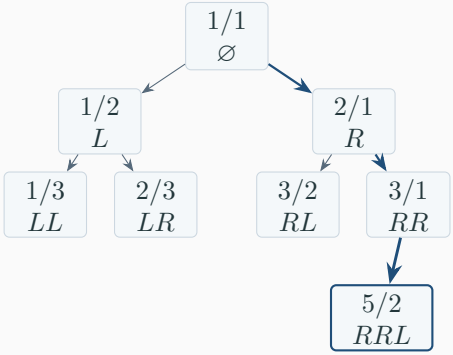
- q-deformations[2] of the modular group and quadratic irrationals.
- A categorical interpretation[3] using the 2-Calabi-Yau category of type A_2.

[2] L. Leclere and S. Morier-Genoud, *q-Deformations in the modular group and of the real quadratic irrational numbers*, Adv. in Appl. Math. **130** (2021), 102223, 28 pp.

[3] A. Bapat, L. Becker, and A. M. Licata, *q-deformed rational numbers and the 2-Calabi-Yau category of type A_2*, Forum Math. Sigma **11** (2023), e47, 41 pp.

Definition of Morier-Genoud–Ovsienko q-rationals

Every positive rational x has a unique Stern–Brocot word U_x in letters L and R, obtained by following the Stern–Brocot tree starting at $1/1$ with transformations $L : x \mapsto x/(x+1)$ and $R : x \mapsto x+1$.



For the word $U_x = \varepsilon_1 \cdots \varepsilon_N$, define

$$[x]_q := \frac{n_x}{d_x}$$

with

$$\begin{pmatrix} n_x \\ d_x \end{pmatrix} := B_{\varepsilon_1}(q) \cdots B_{\varepsilon_N}(q) \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$B_L(q) := \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}, \quad B_R(q) := \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}.$$

Example. For $U_{5/2} = RRL$, we have $\binom{n_{5/2}}{d_{5/2}} = B_R(q)B_R(q)B_L(q)\binom{1}{1}$ and

$$[5/2]_q = \frac{1 + 2q + q^2 + q^3}{1+q}$$

The limit $q \to 1$ gives $[5/2]_q \to 5/2$.

From q-rationals to elliptic rationals

Goal of this talk: Find an elliptic deformation $\boldsymbol{Q_{q,p}(x)}$ of the q-rational construction of Morier-Genoud and Ovsienko:

$$x \in \mathbb{Q}_{>0}, \quad \lim_{p\to 0} Q_{q,p}(x) = [x]_q, \quad \lim_{q\to 1} Q_{q,p}(x) = x.$$

The word "elliptic" means $Q_{q,p}(x)$ should involve elliptic functions with elliptic nome p.

Background of elliptic special functions:

- Elliptic hypergeometric functions appear in quantum integrable systems, algebraic combinatorics, and representation theory.
  For example, Frenkel–Turaev elliptic 6j-symbol (1995):

$${}_{12}V_{11}(a_1; a_6, a_7, \dots, a_{12}; q, p; z) := \sum_{n=0}^{\infty} \frac{\theta(a_1 q^{2n}; p)}{\theta(a_1; p)} \frac{(a_1, a_6, a_7, \dots, a_{12}; q, p)_n}{(q, a_1 q/a_6, a_1 q/a_7, \dots, a_1 q/a_{12}; q, p)_n} (qz)^n$$

$\lim_{p\to 0} {}_{12}V_{11} = {}_{12}\varphi_{11}$, $\lim_{q\to 1} {}_{12}V_{11} = {}_{12}F_{11}$. $\quad \theta(u; p)$: elliptic theta function.

How the project developed

- I got interested in the Morier-Genoud—Ovsienko theory when Ovsienko delivered a talk in the workshop at Reims, French, in June 2018.
  Just after the workshop, I started exploring an elliptic analogue of their theory.
- The initial idea was that one replaces q in $B_L(q) = \left(\begin{smallmatrix} q & 0 \\ q & 1 \end{smallmatrix}\right)$ and $B_R(q) = \left(\begin{smallmatrix} q & 1 \\ 0 & 1 \end{smallmatrix}\right)$ by some elliptic weight $w(q,p)$ such that $\lim_{p\to 0} w(q,p) = q$ and $\lim_{q\to 1} w(q,p) = 1$.
- There were several candidates of such weight $w(q,p)$, including those appearing in
  - Rains' BC_n-symmetric abelian functions (2005),
  - Schlosser's binomial coefficients (2007),
  - Rosengren's connection coefficients (2016), and so on.

  I had some progress using Schlosser's and Rosengren's weights, but could not complete the project.
- On November 1, 2026, I resumed the project with the assistance of AI.
  The project was completed in 5 days, summarized in the preprint
  S.Y., *An elliptic deformation of continued fractions*, arXiv:2611.10104.

Schlosser's elliptic weight

Idea. Use Schlosser's elliptic weight[4] as an elliptic deformation of the base q.

Fix generic $a, b, q \in \mathbb{C}^\times$ and $p \in \mathbb{C}$, $0 < |p| < 1$, let $(u;p)_\infty := \prod_{j=0}^{\infty}(1-up^j)$ and

$$\boldsymbol{\theta(u;p) := (u;p)_\infty (p/u;p)_\infty}, \quad \theta(u_1,\dots,u_k;p) := \prod_i \theta(u_i;p).$$

For $s, t \in \mathbb{Z}_{\geq 1}$, Schlosser's elliptic weight is defined by

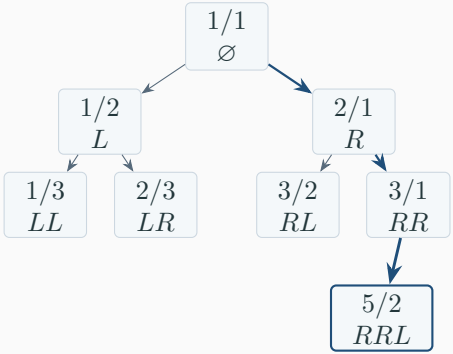
$$\boldsymbol{w_{s,t}} = w_{s,t}(a,b;q,p) := \boldsymbol{q\,\frac{\theta(aq^{s+2t},\, bq^{2s+t-2},\, (a/b)q^{t-s-1};p)}{\theta(aq^{s+2t-2},\, bq^{2s+t},\, (a/b)q^{t-s+1};p)}}.$$

$\lim_{b\to 0}\lim_{a\to 0}\lim_{p\to 0} w_{s,t} = q$ and $\lim_{q\to 1} w_{s,t} = 1$.

[4] M. J. Schlosser, *Elliptic enumeration of nonintersecting lattice paths*, J. Combin. Theory Ser. A **114** (2007), 505–521.

Definition of the elliptic rational number

Stern–Brocot tree with
$L : x \mapsto x/(x+1)$ and $R : x \mapsto x+1$.



For the word $U_x = \varepsilon_1 \cdots \varepsilon_N$ of $x \in \mathbb{Q}_{>0}$, let ℓ_j, r_j count letters L, R in $\varepsilon_1 \cdots \varepsilon_{j-1}$.

Define $\boldsymbol{Q_{a,b;q,p}(x) := N_x/D_x}$,
$\binom{N_x}{D_x} := B_{\varepsilon_1}(\boldsymbol{\omega_1}) \cdots B_{\varepsilon_N}(\boldsymbol{\omega_N})\binom{1}{1}$,
$B_L(\gamma) := \left(\begin{smallmatrix}\gamma & 0\\ \gamma & 1\end{smallmatrix}\right)$, $B_R(\gamma) := \left(\begin{smallmatrix}\gamma & 1\\ 0 & 1\end{smallmatrix}\right)$,
$\boldsymbol{\omega_j = \omega_j(U_x) := w_{1+\ell_j,1+r_j}}$.

Example. $U_{5/2} = RRL$ has weights
$\omega_1 = w_{11}$, $\omega_2 = w_{12}$, $\omega_3 = w_{13}$.
$\binom{N_{5/2}}{D_{5/2}} = B_R(\omega_1)B_R(\omega_2)B_L(\omega_3)\binom{1}{1}$,

$$Q(5/2) = \frac{1 + w_{11} + w_{13} + w_{11}w_{13} + w_{11}w_{12}w_{13}}{1 + w_{13}},$$

$$w_{st} = q\frac{\theta(aq^{s+2t}, bq^{2s+t-2}, (a/b)q^{t-s-1};p)}{\theta(aq^{s+2t-2}, bq^{2s+t}, (a/b)q^{t-s+1};p)}.$$

As $p \to 0$, $a \to 0$, $b \to 0$, we have $w_{st} \to q$,
and $Q(\frac{5}{2}) \to [\frac{5}{2}]_q = (1+2q+q^2+q^3)/(1+q)$.
As $q \to 1$, we have $w_{st} \to 1$, $Q(\frac{5}{2}) \to \frac{5}{2}$.

Basic properties

Proposition 2.5. For every $x \in \mathbb{Q}_{>0}$ successive limits $p \to 0$, $a \to 0$, $b \to 0$ recover the Morier-Genoud–Ovsienko q-rationals:

$$\lim_{b\to 0}\lim_{a\to 0}\lim_{p\to 0} Q_{a,b;q,p}(x) = [x]_q. \qquad \lim_{q\to 1} Q_{a,b;q,p}(x) = x.$$

Proposition 2.6. For every integer $n \geq 1$,

$$Q_{a,b;q,p}(n) = [n]_{a,b;q,p} \coloneqq \sum_{j=0}^{n-1}\prod_{t=1}^{j} w_{1,t}, \quad \text{Schlosser's elliptic integer.}$$

In particular, $\lim_{b\to 0}\lim_{a\to 0}\lim_{p\to 0} Q_{a,b;q,p}(n) = 1 + q + \cdots + q^{n-1} = [n]_q$.

These follow from the definition by straightforward calculations.

Elliptic continued fractions

For rational $x > 1$, write its ordinary regular and negative expansions as

$$x = [a_1, \ldots, a_{2m}] = [\![c_1, \ldots, c_k]\!] \quad (a_i \geq 1,\ c_i \geq 2).$$

Let $v_j \coloneqq \omega_{j+1}(U_x L)$, $t_i \coloneqq \sum_{h<i} a_h$, and $d_i \coloneqq \sum_{h<i}(c_h - 1)$. Put

$$P_t(n) \coloneqq \prod_{j=t}^{t+n-1} v_j, \quad E_t(n) \coloneqq \sum_{i=0}^{n-1} P_t(i), \quad F_t(n) \coloneqq \sum_{i=0}^{n-1} P_t(i)^{-1}.$$

and define the regular/negative elliptic fractions $Q^{\pm}(x)$ as

$$Q^+(x) \coloneqq E_{t_1}(a_1) + \cfrac{P_{t_1}(a_1)}{F_{t_2}(a_2) + \cfrac{P_{t_2}(a_2)^{-1}}{\ddots + \cfrac{P_{t_{2m-1}}(a_{2m-1})}{F_{t_{2m}}(a_{2m})}}}, \quad Q^-(x) \coloneqq E_{d_1}(c_1) - \cfrac{P_{d_1}(c_1 - 1)}{E_{d_2}(c_2) - \cfrac{P_{d_2}(c_2 - 1)}{\ddots - \cfrac{P_{d_{k-1}}(c_{k-1} - 1)}{E_{d_k}(c_k)}}}.$$

Elliptic regular and negative fractions agree

Theorem 3.4. For every rational $x > 1$,

$$\boldsymbol{Q^+(x) = Q^-(x) = Q_{a,b;q,p}(x)}.$$

Example. $5/2 = [2,2] = [\![3,2]\!]$.

$$Q^+(5/2) = 1 + w_{11} + \frac{w_{11}w_{12}}{1 + w_{13}^{-1}}, \quad Q^-(5/2) = 1 + w_{11} + w_{11}w_{12} - \frac{w_{11}w_{12}}{1 + w_{13}}.$$

Both expressions equal $Q(5/2) = \frac{1+w_{11}+w_{13}+w_{11}w_{13}+w_{11}w_{12}w_{13}}{1+w_{13}}$.

Mechanism. Put $K := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $KB_R(\gamma) = B_L(\gamma)$, and we have an identity of *words*

$$B_R^{a_1}(KB_R)^{a_2} \cdots B_R^{a_{2m-1}}(KB_R)^{a_{2m}} = (B_R^{c_1 -1}K) \cdots (B_R^{c_k -1}K)R.$$

We can attach the same ordered weights $v_1, v_2, \ldots$ to the R-letters.

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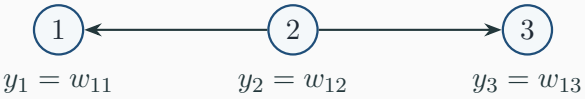
q-degen. Both $Q^{\pm}(x)$ become the Morier-Genoud–Ovsienko regular/negative q-fractions on Slide 1.
Rational Both tend to $2 + 1/2 = 3 - 1/2 = 5/2$.

The type-A quiver of the elliptic rational number

Let $x > 1$ be rational with the Stern–Brocot word $U_x = \varepsilon_1 \cdots \varepsilon_N$. Give vertex i the weight $y_i = \omega_i(U_x)$. An R at position i orients $i - 1 \leftarrow i$, and L orients $i - 1 \rightarrow i$.

The quiver $G_{5/2}$ from $U_{5/2} = RRL$



A closure C is a vertex subset with no arrow leaving it.

$$\mathrm{Cl}(G_{5/2}) = \{\varnothing, \{1\}, \{3\}, \{1,3\}, \{1,2,3\}\}.$$

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q-degen. With all $y_i = q$, its closure polynomial is $1 + 2q + q^2 + q^3$.
Rational With all $y_i = 1$, there are 5 closures (numerator of $5/2$).

Numerator and denominator are F-polynomials

For a quiver G, define its closure polynomial by

$$\boldsymbol{F_G(y) := \sum_{C \in \mathrm{Cl}(G)} \prod_{i \in C} y_i.}$$

Theorem 5.1. For rational $x > 1$, let G_x be its type-A quiver, and G'_x the suffix quiver obtained by removing the first a_1 vertices. ($x = [a_1, \ldots, a_{2m}]$.)

$$\boldsymbol{N_x = F_{G_x}(y), \quad D_x = F_{G'_x}(y).}$$

Example. For $x = 5/2 = [2, 2]$, we have $G_{5/2} = (1) \leftarrow (2) \rightarrow (3)$, $G'_{5/2} = (3)$, and

$$N_{5/2} = F_{G_{5/2}}(y) = 1 + y_1 + y_3 + y_1y_3 + y_1y_2y_3,$$
$$D_{5/2} = F_{G'_{5/2}}(y) = 1 + y_3.$$

These are quiver F-polynomials for thin identity-arrow representations.

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q-degen. $y_i \to q$ gives $N_{5/2}(q) = 1 + 2q + q^2 + q^3$, $D_{5/2}(q) = 1 + q$.
Rational $y_i \to 1$ gives $N_x \to n$ and $D_x \to d$ for $x = n/d$.

Summary

1. **Construction.** Put Schlosser's elliptic weights on Stern–Brocot word.
2. **Algebra.** Regular and negative elliptic continued fractions agree.
3. **Combinatorics.** Numerator and denominator are weighted closure polynomials of type-A quivers.

The two degenerations recover the known theory and the original rational:

$$Q_{a,b;q,p}(x) \xrightarrow[\text{Y.}]{p,a,b\to 0} [x]_q \xrightarrow[\text{Morier-Genoud–Ovsienko}]{q\to 1} x.$$

We also have elliptic counterpart of:

- Input-shift ($x \mapsto x+1$, $x \mapsto \frac{x}{x+1}$) formulas (Proposition 2.6).
- Farey sum ($(n/d) \oplus (n'/d') := (n+n')/(d+d')$) recursions (Theorem 4.1).
- Inversion ($p \mapsto p^{-1}$) formula (Proposition 4.5).
- Positivity in the letters $w_{s,t}$ (Proposition 4.7).
- Toward interpretation via elliptic cohomology (§6, not completed topic).

Thank you.