Approximate Majorization and High-Order Capacity of Quantum Depolarizing Channels

arXiv:2609.37240v1

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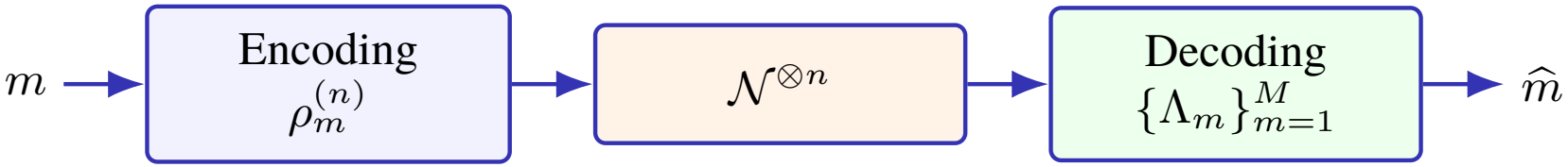
Nagoya–Shenzhen Workshop 2026

Quantum Channel Coding



A central problem is to determine the classical capacity of quantum channels.

$$C(\mathcal{N}) := \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log M(n,\varepsilon) \underset{\left[\begin{array}{c}\text{Holevo, 1998;}\\ \text{Schumacher–Westmoreland, 1997}\end{array}\right]}{\overset{\text{HSW}}{=\!=\!=}} \lim_{n\to\infty} \frac{1}{n}\chi(\mathcal{N}^{\otimes n}).$$

- **Operational meaning:** $M(n,\varepsilon)$ is the maximum size of a length-n code for $\mathcal{N}$ with average error probability $\varepsilon \in (0,1)$.

$$\underbrace{\chi(\mathcal{N})}_{\text{Holevo information}} = \max_{\{p_x,\rho_x\}} \left[S\left(\sum_x p_x \mathcal{N}(\rho_x)\right) - \sum_x p_x S(\mathcal{N}(\rho_x)) \right].$$

Not additive in general [Hastings, 2009] NP-hard to compute [Beigi–Shor, 2008]

In practice, however, the number of channel uses is **finite**.

First-order asymptotics [Cheng–Tomamichel–26]

For any finite-dimensional memoryless quantum channel $\mathcal{N}$ and every fixed $\varepsilon \in (0,1)$, the strong converse gives

$$\lim_{n\to\infty} \frac{1}{n} \log M(n,\varepsilon) = C(\mathcal{N}).$$

Even if the capacity is known, we still need the higher-order terms.

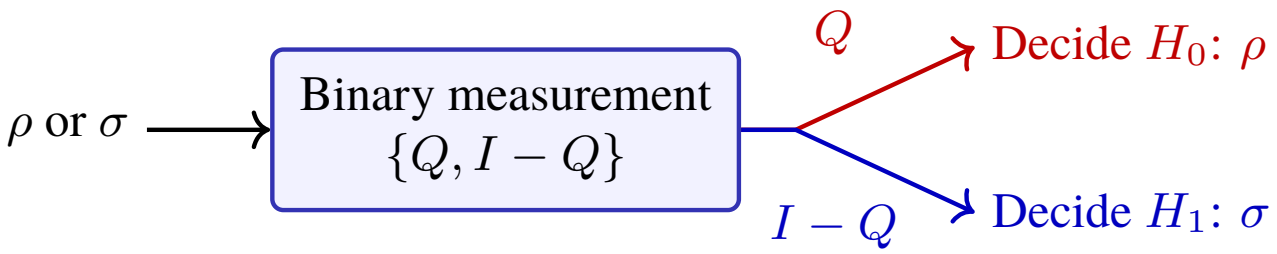
$$\log M(n,\varepsilon) = \underbrace{nC(\mathcal{N})}_{\text{First order}} + \underbrace{o(n)}_{\text{Higher-order}} \quad ?\ .$$

Questions at finite blocklength:

- How close can finite-length codes get to capacity?
- How much can entangled codewords and collective decoding help?

Quantum Hypothesis Testing

Two hypotheses: $H_0 : \rho$ (null) $\qquad$ $H_1 : \sigma$ (alternative).



Type-I error

True $\rho \longrightarrow$ decide σ.

$\alpha(Q) := \mathrm{Tr}[\rho(I - Q)]$.

Type-II error

True $\sigma \longrightarrow$ decide ρ.

$\beta(Q) := \mathrm{Tr}(\sigma Q)$.

Hypothesis-testing divergence: Minimize type-II error subject to type-I error $\leq \varepsilon$.

$$D_H^\varepsilon(\rho\|\sigma) := -\log \min_{\substack{0\leq Q\leq I \\ \alpha(Q)\leq\varepsilon}} \beta(Q).$$

Data Processing Inequality: $D_H^\varepsilon(\Phi(\rho)\|\Phi(\sigma)) \leq D_H^\varepsilon(\rho\|\sigma)$.

From Channel Coding to Hypothesis Testing

Outputs: $X_m := \mathcal{N}^{\otimes n}(\rho_m^{(n)})$ on $\mathbb{C}^D$; decoder POVM: $\{\Lambda_m\}$.

Channel coding: Maximize M subject to

$$P_{\text{err}} = 1 - \frac{1}{M}\sum_m \text{Tr}(\Lambda_m X_m) \le \varepsilon \qquad \sum_m \Lambda_m = I_D$$

$$\Omega := \frac{1}{M}\sum_m |m\rangle\langle m| \otimes X_m \qquad \Theta := \frac{I_M}{M} \otimes \frac{I_D}{D}$$

Binary measurement $\{T, I - T\}$

$$T := \sum_m |m\rangle\langle m| \otimes \Lambda_m$$

$$\alpha(T) \le \varepsilon \qquad \beta(T) = \frac{1}{M}$$

optimize the test

Hypothesis-testing meta-converse [Wang-Renner-12]

$$\log M = -\log \beta(T) \le D_H^{\varepsilon}(\Omega \| \Theta)$$

Depolarizing channel:

$$\mathcal{D}_{p,d}(\rho) = (1-p)\rho + \frac{p}{d} I_d, \qquad 0 \le p \le \frac{d^2}{d^2-1}.$$

King's theorem [King-03]

$$\left\|\mathcal{D}_{p,d}^{\otimes n}(\rho^{(n)})\right\|_q \le \left\|\mathcal{D}_{p,d}(|0\rangle\langle 0|)^{\otimes n}\right\|_q, \quad q \ge 1 \implies \begin{array}{c}\textbf{Minimum output entropy additivity} \\ \textbf{Holevo additivity}\end{array} \implies \textbf{Classical capacity } C(\mathcal{D}_{p,d})$$

Third-order achievability: **Orthogonal-basis encoding + matching product measurement**

Classical d-ary symmetric channel

$$\log M(n,\varepsilon) \ge nC(\mathcal{D}_{p,d}) + \sqrt{nv_{d,p}}\,\underbrace{\Phi^{-1}(\varepsilon)}_{\text{standard normal quantile}} + \frac{1}{2}\log n + O(1), \qquad (v_{d,p} > 0).$$

[Polyanskiy-Poor-Verdú-10]

High-order upper bound: need a stronger spectral ordering beyond King's norm inequalities.

Perhaps Majorization?

Majorization

For vectors $x, y \in \mathbb{R}^D$, let $x^{\downarrow}$ and $y^{\downarrow}$ denote their entries in nonincreasing order.

$$x \prec y \quad \Longleftrightarrow \quad \begin{cases} \displaystyle\sum_{j=1}^{k} x_j^{\downarrow} \le \sum_{j=1}^{k} y_j^{\downarrow}, & 1 \le k < D, \\ \displaystyle\sum_{j=1}^{D} x_j = \sum_{j=1}^{D} y_j . \end{cases}$$

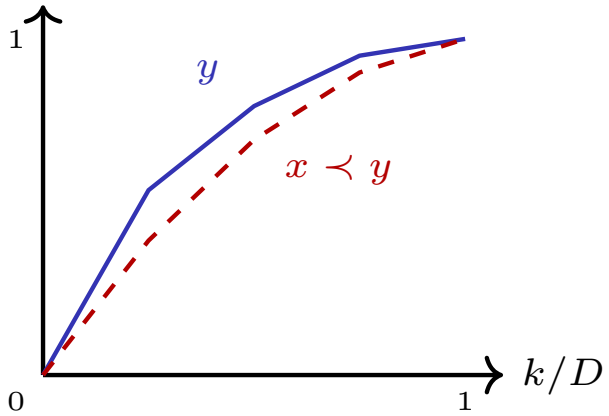
- $x \prec y$ means that x is more spread out, while y is more concentrated.

$$\underbrace{\left(\frac{1}{D}, \ldots, \frac{1}{D}\right)}_{\text{uniform}} \prec p \prec \underbrace{(1, 0, \ldots, 0)}_{\text{concentrated}} .$$

- **Partial order** up to permutations.

Example: probability vectors

cumulative mass



For Hermitian matrices A, B, we say $A \prec B$ if their eigenvalues satisfy $\lambda^{\downarrow}(A) \prec \lambda^{\downarrow}(B)$.

Uhlmann's theorem

$A \prec B$ if and only if $A = \Phi(B)$ for a **mixed-unitary** (also unital) channel Φ.

Majorization controls a whole family of spectral functionals at once.

$$A \prec B \quad \Longrightarrow$$

Schur-convex / Schur-concave

$$\begin{aligned} &\text{Schatten norms:} && \|A\|_p \le \|B\|_p, \quad p > 1, \\ &\text{Rényi entropies:} && H_\alpha(A) \ge H_\alpha(B), \quad \alpha > 0. \\ &&& \vdots \end{aligned}$$

Individual spectral inequalities do not, in general, imply majorization.

For any input $\rho^{(n)}$, write $X_n := \mathcal{D}_{p,d}^{\otimes n}(\rho^{(n)}), \quad Y_n := \left[\mathcal{D}_{p,d}(|0\rangle\langle 0|)\right]^{\otimes n}$.

Conjecture $\lambda^{\downarrow}(X_n) \prec \lambda^{\downarrow}(Y_n), \quad \forall\, \rho^{(n)}$

What does it mean?

$$\log M(n,\varepsilon) \le \sup_{\rho^{(n)}} D_H^{\varepsilon}\left(X_n \middle\| \frac{I_{d^n}}{d^n}\right) \qquad \text{Meta-converse}$$

$$= \sup_{\rho^{(n)}} D_H^{\varepsilon}\left(\Phi(Y_n) \middle\| \Phi\left(\frac{I_{d^n}}{d^n}\right)\right) \qquad \text{Uhlmann}$$

$$\le D_H^{\varepsilon}\left(Y_n \middle\| \frac{I_{d^n}}{d^n}\right) \qquad \text{DPI}$$

$$= nC(\mathcal{D}_{p,d}) + \sqrt{nv_{d,p}}\,\Phi^{-1}(\varepsilon) + \frac{1}{2}\log n + O(1) \qquad \text{Classical i.i.d. testing}$$

Directly match the achievability bound!

Still open (even with GPT-6 Astra)

Can approximate majorization be precise enough to determine the higher-order terms?

$$\delta_{\text{maj}}(X_n, Y_n) := \max_{1 \le k \le D} \left\{ \sum_{j=1}^{k} \lambda_j^{\downarrow}(X_n) - \sum_{j=1}^{k} \lambda_j^{\downarrow}(Y_n) \right\}.$$

$\uparrow$

Measuring the largest violation of partial-sum inequalities.

$$\delta_{\text{maj}}(X_n, Y_n) \le \delta_n \quad \text{Ky Fan principle} \Longrightarrow \quad D_H^{\varepsilon}\left(X_n \middle\| \frac{I_D}{D}\right) \le D_H^{\varepsilon+\delta_n}\left(Y_n \middle\| \frac{I_D}{D}\right).$$

$\uparrow$

$$nC(\mathcal{D}_{p,d}) + \sqrt{nv_{d,p}}\,\Phi^{-1}(\varepsilon + \delta_n) + \frac{1}{2}\log n + O(1)$$

How to obtain an approximate majorizing relation?

From Shifted Log-Det Ordering to Majorization

Define $L_t(X) := \ln\det(tI + X)$ for $t > 0$.

Main Theorem

Fix a state σ whose nonzero eigenvalues are not all equal. For any sequence of states $\{\rho^{(n)}\}$ satisfying $L_t(\rho^{(n)}) \geq L_t(\sigma^{\otimes n})$ for every $t > 0$, we have

$$\delta_{\text{maj}}(\rho^{(n)}, \sigma^{\otimes n}) = O\left(\frac{\log\log n}{\sqrt{n}}\right).$$

Key idea

- **General bound.** For states X, Y satisfying $L_t(X) \geq L_t(Y)$ for every $t > 0$,

$$\delta_{\text{maj}}(X, Y) \leq q(Y)\Big[1 + 2\ln\big(1 + \ln(e/q(Y))\big)\Big].$$

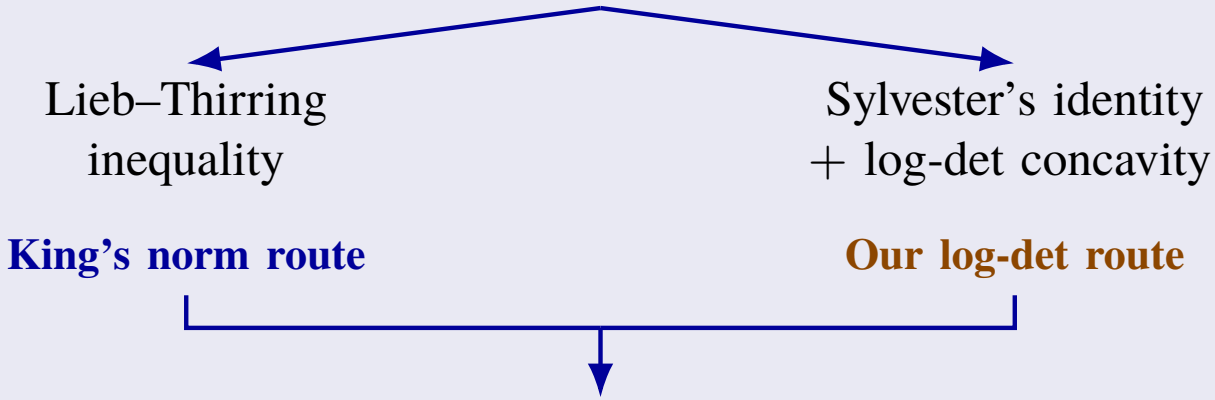
  Here $q(Y) := \sup_{u>0} \sum_{j:\, u/2 \leq \lambda_j(Y) < u} \lambda_j(Y)$ measures spectral concentration.

- **i.i.d. reference.** For $Y_n = \sigma^{\otimes n}$, spectral anti-concentration gives

$$q(Y_n) = O(n^{-1/2})$$

Main Theorem: $L_t(X_n) \geq L_t(Y_n), \quad \forall\, t > 0.$

Control off-diagonal coherence of arbitrary inputs



$$X_n \prec Y_n \implies \underset{\forall\, t>0}{L_t(X_n) \geq L_t(Y_n)} \implies \underset{\forall\, q\in(1,2]}{\|X_n\|_q \leq \|Y_n\|_q} \overset{q\downarrow 1}{\Longrightarrow} S(X_n) \geq S(Y_n)$$

Summary of the Results

Log-det comparison

$$L_t(X_n) \geq L_t(Y_n)$$

Approximate majorization

$$\delta_n(X_n, Y_n) = O\left(\frac{\log\log n}{\sqrt{n}}\right)$$

Normal quantile expansion

$$\Phi^{-1}(\varepsilon + \delta_n) = \Phi^{-1}(\varepsilon) + O(\delta_n)$$

$$\log M(n, \varepsilon) = nC(\mathcal{D}_{p,d}) + \sqrt{nv_{d,p}}\,\Phi^{-1}(\varepsilon) + \frac{1}{2}\log n + O(\log\log n).$$

Improvement from entangled inputs and collective decoding

A rare characterization in quantum channel theory.

Thanks for your attention!